1. Plasma characteristics (discussion and correction: 5.11.2013)

A) What is a ideal/thermal/degenerated plasma? Explain the concept of Debye shielding!

B) Which of the plasmas below are ideal? Calculate the Debye length and the number of particles in the Debye sphere!

fusions plasma (magnetic confinement): $n = 10^{20} m^{-3}, T = 10keV$
fusions plasma (intertial confinement): $n = 10^{31} m^{-3}, T = 10keV$
ionosphere:$n = 10^{11} m^{-3}, T = 0.05eV$
flame:$n = 10^{14} m^{-3}, T = 0.1eV$
solar corona: $n = 10^{12} m^{-3}, T = 100eV$
core of the sun: $n = 10^{31} m^{-3}, T = 1keV$
electron gas in a metal: $n = 10^{29} m^{-3}, T = 300K$

C.1) Show that the Debye-Hueckel potential with an external charge q0 at r = 0,

$$\phi(r) = q_0/(4\pi\epsilon_0 r) exp(\sqrt{2}r/\lambda_D)$$

is a solution of the Poisson equation in spherical coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\phi(r)}{\partial r}) = \frac{2e^2 n_0 \phi(r)}{\epsilon_0 k_B T}$$

C.2) By integrating the plasma charge density e (ni - ne) over the entire space, show that the total charge is equal to -q0. Use the linearised Boltzmann factor $1 - e\phi/(k_B T)$ to calculated the densities.

C.3) Derive the same result by using Gauss' law. To this end, calculate the electric flux through a spherical surface in the limit $r \to \infty$.

With additional literature (e.g. Kaufmann, Plasmaphysik und Fusionsforschung; Fliessbach, Allgemeine Relativitätstheorie, Spektrum):
D*) What is a ideal Fermi gas and the Fermi energy? What is the typical screening length ("Thomas-Fermi-length") in a degenerated electron plasma?
E*) What is the Chandrasekhar limit for a white dwarf?

Solution, first sheet (22.10. and 5.11.2013)

A) see lecture

B) The debye length is defined as :

$$\lambda_d = \sqrt{\frac{\epsilon_0 k_B T}{e^2 n}}$$

Since all temperatures are given in eV or keV we formally set $k_B = 1$ and arrive at

$$\lambda_d = \sqrt{\frac{\epsilon_0 \cdot 1000 \cdot T[keV]}{en[m^-3]}} = 2.35 \cdot 10^5 \sqrt{\frac{T[keV]}{n[m^-3]}} m$$

$$N_d = n_e \frac{4\pi}{3} \lambda_D^3$$

The Debye lengths and the number of particles in the for the plasmas given are:
Fusion, magnetic: $\lambda_d = 7.4 \cdot 10^{-5} m, N_d = 1.7 \cdot 10^8$
Fusion, inertial: $\lambda_d = 2.35 \cdot 10^{-10} m, N_d = 5.4 \cdot 10^2$
ionophere: $\lambda_d = 5.3 \cdot 10^{-3} m, N_d = 6.1 \cdot 10^4$
flame: $\lambda_d = 2.4 \cdot 10^{-4} m, N_d = 5.4 \cdot 10^3$
solar corona: $\lambda_d = 7.4 \cdot 10^{-2} m, N_d = 1.6 \cdot 10^9$
solar core: $\lambda_d = 7.4 \cdot 10^{-11} m, N_d = 17$
electron gas: use Thomas Fermi length (see problem D): $\lambda_{TF} = \frac{1}{\sqrt{2}} (\frac{\pi}{4})^{1/6} n_e^{-1/6} \sqrt{a_0}$ with a_0 as Bohr's radius $a_0 = 5.3 \cdot 10^{-11} m$. For the given n_e we have $\lambda_{TF} = 0.76 \mathring{A}$, $N_d = 0.185$.

In order to directly see, if a plasma is ideal or not, one has to compare electrostatic energy with kinetic energy. The plasma is ideal if the kinetic energy is much larger than the potential energy. This leads to

$$\frac{T[eV]}{n[m^{-3}]^{1/3}} > 10^{-9}$$

All plasmas given above are ideal except the electron gas in a metal.

C1) Calculate radial derivatives:

$$\frac{\partial \phi}{\partial r} = -\frac{q_0}{4\pi\epsilon_0 r} e^{(-\sqrt{2}r/\lambda_D)} (1/r + \sqrt{2}/\lambda_D)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\phi}{\partial r}) = ... = \frac{q_0}{4\pi\epsilon_0 r}e^{(-\sqrt{2}r/\lambda_D)}\cdot\frac{2}{\lambda_d^2} = \frac{2e^2ne\phi}{\epsilon_0 k_B T}$$

C2) The charge density is given for $r \neq 0$ by

$$\varrho = -\epsilon_0\Delta\phi(r)$$

with $\phi(r) = \frac{q_0}{4\pi\epsilon_0 r}e^{-\sqrt{2}r/\lambda_D}$
We've just calculated (C1):

$$\varrho = -\epsilon_0\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\phi}{\partial r}) = -\frac{q_0}{2\pi r\lambda_D^2}e^{-\sqrt{2}r/\lambda_D}$$

or directly starting from the densities (as in the lecture), we use that

$$\varrho = -e(n_e - n_i) = -n_{e0}(1+e\phi/(k_BT)) - (n_{i0}(1-e\phi/(k_BT))) = -2e^2n_{e0}\phi/(k_BT) = -\frac{q_0}{2\pi r\lambda_D^2}e^{-\sqrt{2}r/\lambda_D}$$

Integrating over the total volume gives:

$$Q = \int\varrho dV = \int_0^\infty\varrho(r)4\pi r^2dr = -q_0$$

using $\int_0^\infty xe^{-ax}dx = 1/a^2$

C3) The electric flux $\psi = \int\mathbf{E}\cdot\mathbf{dA}$ vanishes at infinity for the given potential since the exponential function drops off faster than any power of r:

$$\int\mathbf{E}\cdot\mathbf{dA} = 4\pi\frac{q_0}{4\pi\epsilon_0 r}e^{(-\sqrt{2}r/\lambda_D)}(1/r + \sqrt{2}/\lambda_D) \rightarrow 0$$

We used from C1:

$$E = -\nabla\phi = \frac{q_0}{4\pi\epsilon_0 r}e^{(-\sqrt{2}r/\lambda_D)}(1/r + \sqrt{2}/\lambda_D)$$

Since

$$\int\mathbf{E}\cdot\mathbf{dA} = \int\nabla\cdot\mathbf{E}dV = \frac{1}{\epsilon_0}\int\varrho_{total}dV \rightarrow 0$$

where ϱ_{total} is the total charge in the volume, i.e. the external charge $\delta(r=0)\cdot q_0$ and the charge of the shielding cloud. Since according to the last formula the total charge also goes to 0 for large radii, the charge of the cloud has to compensate the external charge at $r = 0$ and therefore $\varrho_{cloud} = -q_0$