Collisions (discussion and correction: 3.12.2013)

1.Explain why there is a difference for the collision times of electrons and ions for the momentum and energy exchange (elastic collisions).

2. The Coulomb Logarithm

Λ is the ratio between small-angle and large-angle scattering processes inside a plasma. Calculate this quantity and the Coulomb logarithm $\ln \Lambda$ for a laboratory, a fusion and a laser plasma with the parameters $T_e = 2/10.000/1.000$ eV and $n_e = 10^{17}/10^{21}/10^{27}$ m^{-3}, respectively .

3a. Show that the total cross section (integral over all possible impact parameters) for Coulomb collisions diverges!

3b. Derive an expression for the force that an ensemble of test particles (initial velocity $v_z \mathbf{e}_z$) experiences due to static background particles (Coulomb collisions)!

Hint: First, calculate the change in velocity along the initial direction $\mathbf{e}_z$ as function of the scattering angle (trigonometric analysis). Then use that the total force in that direction is equal to the change in momentum per time and integrate over all scattering angles.

3c. How does the calculation in 3b change, if shielding is taken into account?

3d. Using the results above, derive an expression for the resistivity of a plasma!

1. (see lecture) in short: since the momentum exchange for light particles (electrons) scattering on heavy particles (ions) is much larger than vice versa, there is a difference between τ_{ei} and τ_{ie}. The difference between electron-ion, electron-electron and ion-ion collisions is due to the fact that the reduced mass of the system is different: $m_{red} = m_1 \cdot m_2/(m_1 + m_2)$. $m_{e,i} = m_e$, $m_{e,e} = m_e/2$, $m_{i,i} = m_i/2$. Further, there is a dependence of the scattering cross section on the charge number $\sim Z^2$. (Note that often one power of Z disapperas when $n_e = Zn_i$ is used .)
For the energy exchange the asymmetry between el-ion and ion-el collisions disappears. Exchange processes (temperature equilibriation) happen fastest between the same species (ion-ion, el-el):

$$\frac{\tau_{ee}^E}{\tau_{ii}^E} = Z_i^3 \sqrt{\frac{m_e}{m_i}}$$

$$\frac{\tau_{ee}^E}{\tau_{ei}^E} = 2\sqrt{2} Z_i \frac{m_e}{m_i}$$

Note that for detailed calculations one needs to take into account distribution functions for the background species - details see second semester!

2. The Coulomb logarithm measures the importance of small and large angle scattering events (assuming deuterium):

$$b_{90} = \frac{e^2}{4\pi\epsilon_0 3k_B T} = \frac{1}{12\pi\lambda_D n}$$

with

$$\lambda_D = \sqrt{\frac{\epsilon_0 k_B T}{e^2 n}}$$

Coulomb logarithm:

$$\Lambda^2 \approx \frac{\lambda_D^2}{b_{90}^2}$$

$$\Lambda = 12\pi(\epsilon_0/e)^{3/2}(k_B T/e)^{3/2} n^{-1/2}$$

$$\Lambda = 15.5 \cdot 10^{12} T[eV]^{3/2} n[m^{-3}]^{-1/2}$$

$$n = 1 \cdot 10^{17}; \quad T = 2eV; \quad \ln\Lambda = 11.85 \;\; (ln(0.14 \cdot 10^6))$$

$$n = 1 \cdot 10^{21}; \quad T = 10000eV; \quad \ln\Lambda = 20 \;\; (ln(490 \cdot 10^6))$$

$$n = 1 \cdot 10^{27}; \quad T = 1000eV; \quad \ln\Lambda = 9.65 \;\; (ln(0.016 \cdot 10^6))$$

3.a

The differential cross section is given (mechanics):

$$\frac{d\sigma}{d\Omega} = \alpha^2 \Big[\frac{1}{2mv^2 \sin^2 \chi/2}\Big]^2$$

where the Coulomb potential is

$$U = -\alpha/r \quad \text{with} \quad \alpha = \frac{e \cdot Ze}{4\pi\epsilon_0}$$

and

$$\tan(\chi/2) = b_{90}/b$$

angle between to cones of angle χ and $\chi + d\chi$:

$$d\Omega = 2\pi \sin(\chi) d\chi$$

$$\sigma = \int_0^\infty \frac{d\sigma}{d\Omega} d\Omega = \int_{\chi_{min}}^\pi \frac{c \cdot \sin\chi}{\sin^4(\chi/2)} d\chi = c \cdot \Big[\frac{-2}{\sin^2 \chi/2}\Big]_0^\pi \to \infty$$

for $\chi \to 0$.

3b. The total friction force in direction z is a sum over the forces on all the incindent test particles:

$$R = m\frac{d}{dt}\sum \delta v_z$$

The change of the velocity in z direction is given by $\delta v_z = -2v \sin^2 \chi/2$ where v is the initial velocity and χ the scattering angle (trigonometric considerations, see figure below). Then, express sum as integral over all scattering angles and use that $dN/dt = nvd\sigma$:

$$R = m\frac{d}{dt}\sum \delta v_z = m\int_0^\infty \delta v_z \frac{dN}{dtd\sigma} d\sigma = -m\int_0^\infty \frac{2v \sin^2(\chi/2) nv\alpha^2}{(2mv^2)^2 \sin^4(\chi/2)} d\Omega = \frac{2\alpha^2 mnv^2}{4m^2v^4}\int \frac{2\pi \sin\chi d\chi}{\sin^2(\chi/2)}$$

$$R = \frac{\pi\alpha^2 n}{mv^2}\int_{chi_{min}}^\pi \frac{\sin\chi d\chi}{\sin^2(\chi/2)} = \frac{4\pi\alpha^2 n}{mv^2} \ln(\sin(\chi/2))|_0^\pi$$

Again, $R \to -\infty$ for small χ.

3c. If shielding is taken into accout, one only integrates to λ_D, since the charge is assumed to be shielded for very small scattering angles. For small angles (ideal plasma condition!) where $\lambda_D \gg b_{90}$ we can use $\tan x \approx x$:

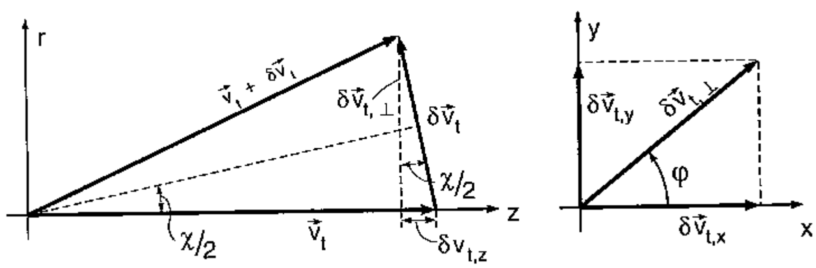
$$\ln(\sin(\chi/2)) = -\ln \frac{\lambda_D\sqrt{1 + \frac{b_{90}^2}{\lambda_D^2}}}{b_{90}} \approx -\ln\frac{\lambda_D}{b_{90}} = -\ln\Lambda$$

using $sin^2 x = tan^2 x/(1 + tan^2 x)$ and $tan(\chi/2) = b_{90}/b$.

$$|\delta\vec{v}_t| = 2v_t sin(\chi/2) \qquad |\delta\vec{v}_{t,\perp}| = 2v_t sin(\chi/2)cos(\chi/2)$$

$$\delta v_{t,z} = -2v_t sin^2(\chi/2) = -2v_t \frac{tg^2(\chi/2)}{tg^2(\chi/2)+1} = -2v_t \frac{s_\perp^2}{s_\perp^2+s^2}$$

$$\delta v_{t,x} = 2v_t \frac{ss_\perp}{s_\perp^2+s^2} cos\varphi \qquad \delta v_{t,y} = 2v_t \frac{ss_\perp}{s_\perp^2+s^2} sin\varphi$$



3d. In the stationary limit we set the electric force eE equal to the momentum loss of the electrons per time:

$$eE = m_e v_e / \tau_{e,i}$$

For $\tau_{e,i,}$ we use the momentum exchange time (see lecture):

$$\tau_{e,i,} = \frac{KT_e^{3/2}}{\ln \Lambda Z n_e}; \quad \text{with} \quad K = \frac{\sqrt{3}\cdot 6\pi\epsilon_0^2\sqrt{m_e}}{e^4}$$

Using $j = \sigma E$ and $j = en_e v_e$ we obtain for σ:

$$\sigma = \frac{e^2 K T_e^{3/2}}{m_e Z \ln \Lambda}$$

The conductivity increases with temperature, is independent (only via the Coulomb logarithm) of the density. ($\eta = 1/\sigma$).
The same result can be obtained if we directly set $eE = R$ with R the force as calculated in 3b and 3c.