4. Ambipolarity, flourescent lamp, optical thickness (discussion and correction: 17.12.2013)

1. Explain 'ambipolar diffusion' for a low temperature plasma ($T_e \gg T_i$)

2. Calculate the radial density distibution of electrons in an infinitely long flourescent lamp (low pressure,$T_e \gg T_i \gg T_{neutral\ gas}$) with the cross section radius R. Because of the very small densities, the charged particles are not annihilated via recombination or collisions but via ambipolar radial diffusion. Assume that the electric field along the axial and radial direction is constant.
Use the following ansatz: in equilibrium the production rate of electrons has to balance the loss via ambipolar diffusion. This loss is the divergence of the ambipolar current $j_{diff} \sim -D_a \nabla n_e$ (see exercise 1). Assume that the production rate is of the form: $R_N = e \cdot n_e \cdot const_{neutral\ gas}$. One arrives at a differential equation for n_e in form of Bessel's differential equation. Use the boundary conditions $n_e(R) = 0$ and $n_e(0) = n_0$ (and the fact that the Besselfunction crosses 0 for $x = 2.405$ for the first time) for calculating T_e via the Einstein relation for D_a.

3. If one measures the emission lines of a plasma the self-absorption of the plasma perturbs the results. Therefore, it is important to confirm that the plasma is optically thin for the range of wave lengths under consideration. In order to check this, one can use the following experimental setup: a homogeneous plasma in y-z direction (sufficiently large) is confined between $x = 0$ and $x = x_0$. At x_0 we place our detector. We carry out two measurements of the spectral intensity: once with and once without a mirror at $x = 0$ that is assumed to reflect perfectly the range of wave lengths under consideration. Calculate the ratio of the two measurements in dependence of the optical thickness τ for a plasma with a Boltzmann distribution of the excited states.

Solution

1. see lecture

2. In a stationary situation the production rate has to balance the loss rate:

$$R_{loss} = R_{prod}$$

The losses are described by ambipolar radial diffusion, i.e. the divergence of the ambipolar current $j = e\Gamma_a$ with $\Gamma_a = -D_a \nabla n$:

$$R_{loss} = e\ \nabla \cdot (-D_a \nabla n_e) = -e\ D_a \frac{1}{r}\frac{d}{dr}(r\frac{dn_e}{dr})$$

Together with the ansatz given for the production rate $R_N = e \cdot n_e \cdot const_{neutral\ gas} \equiv e \cdot n_e \cdot c$ one arrives at the following differential equation:

$$-D_a n_e'' - D_a \frac{n_e'}{r} - c \cdot n_e = 0$$

Introducing $x = r\sqrt{\frac{c}{D_a}}$ and using $\frac{dx}{dr} = \sqrt{c/D_a}$, one can rewrite this equation as:

$$x^2 n_e'' + x \cdot n_e' + (x^2 - 0^2) n_e = 0$$

The two linear independent solutions of this equation of second order are:

$$n_e = c_1 J_0(x) + c_2 Y_0(x)$$

with

$$J_0(x) = \sum_{t=0}^{\infty} \frac{(-1)^t (x/2)^{2t}}{\Gamma(t+1) t!}$$

Γ here is the gamma function $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$.

$$Y_0(x) = lim_{p \to 0} \frac{J_p(x) \cos \pi p - J_{-p}(x)}{\sin \pi p}$$

Now we use the given boundary conditions: since the on-axis density has to be finite ($n_e(0) \equiv n_0$) the coefficient c_2 has to vanish: $Y_0(x)$ diverges for $x \to 0$. Therefore, the solution is:

$$n_e(r) = n_0 J_0\Big(r\sqrt{\frac{c}{D_a}}\Big)$$

The other boundary condition $n_e(r) = 0$ gives a constraint on the argument: the first 0 of the Bessel function is at $x = 2.405$ and therefore:

$$2.405 = R\sqrt{\frac{c}{D_a}}$$

For $T_e \gg T_i$, $D_a = D_i \frac{T_e}{T_i}$ and using the Einstein relation $D_i = \frac{\mu_i k_B T_i}{e}$ we obtain:

$$D_a = k_b T_e \mu_i / e$$

and therefore:

$$T_e = \frac{R^2 \cdot c \cdot e}{2.405^2 k_B \mu_i}$$

Using that $\mu_i \sim 1/n_n$ and that $c = n_n \cdot \langle v_e \cdot e \rangle_{T_e}$ one finds:

$$T_e \sim R^2 n_n^2$$

with n_n the neutral gas density. The result means that the electron temperature is determined by the radius and the neutral density of the filling gas. Together with the dependence of the voltage U on $p \cdot d$ (pressure times length of the tube) one can determine all relevant parameters for the lamp.

3. A mirror at $x = 0$ generates for positive x a radiation field that is equivalent to the radiation of a field between $-x_0$ and 0. The optical thickness just doubles according to its definition:$\tau = \int_x^{x_0} \alpha' dx$. If we measure the radiation at x_0 without mirror, the spectral density is $L_\nu = B_\nu(T)(1 - e^{-\tau})$. With the mirror, it is $L_\nu = B_\nu(T)(1 - e^{-2\tau})$. The ratio is

$$Q = \frac{(1 - e^{-2\tau})}{(1 - e^{-\tau})} = \frac{(1 + e^{-\tau})(1 - e^{-\tau})}{(1 - e^{-\tau})} = 1 + e^{-\tau}$$

For small τ we have $Q = 2 - \tau$ independent from the observation angle and detector sensitivity. Small values of τ that disturb the measurements can be determined and the offset can be takes into account: if one scans the detected frequency, one sees in an optically thin plasma s small scattering around 2 (usually within the sensitivity of the measurement) and in the line centres a reduction that allows one to determine τ.