7. Fluid drifts (discussion and correction: 4.2.2014)

1. Consider an cylidrical plasma (oriented along the z-axis) with radius $R = 1cm$ in a homogeneous magnetic field with $\mathbf{B} = (0, 0, B_{z0})$ with $B_{z0} = 4T$. Radially it has the following temperature and density distribution:

$$T_e(r) = T_i(r) = 0.25eV$$

and

$$n_{i,e}(r) = n_0 \cdot (1 - r^2/R^2)$$

where $n_0 = 1.0 \cdot 10^{16} m^{-3}$. Due to the density gradient there is a diamagnetic current j_{dia} that reduces the magnetic field. Calculate $B_z(r)$ and the reduction of the magnetic field in the plasma centre $dB = B_{z0} - B_z$!

2. A cylindrical plasma (radius a) in a homogeneous magnetic field has parabolic radial temperature and pressure profiles with the central values $T_{e.i} = 100eV$ and $n_{e,i} = 10^{20} m^{-3}$, respectively. In addition, there exists a purely radial electric field, that has a value of $E_r = 1kV/m$ at mid radius $(a/2)$.

Use the stationary equation of motion $(d\mathbf{u}/dt = 0)$ in the two fluid picture to answer the following questions:
a) What is the general equation for the fluid velocity in the poloidal direction (u_θ)?
Hint: multiply the equation of motion with $\times\mathbf{B}$ and solve for u_θ.
b) Compare the resulting terms with the single-particle drifts!
c) At $r = a/2$, what are the resulting values and directions of the drifts of electrons and ions that relate to the terms with E_r and the pressure gradient $\partial p/\partial r$?

3. Vlasov equation

Show that the Maxwell-Boltzmann distribution

$$f(v) = \left(\frac{m}{2\pi T}\right)^{3/2} \exp\left[\frac{-(mv^2/2 + q\phi)}{T}\right]$$

is a solution of the Vlasov equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} f + \frac{q\mathbf{E}}{m} \cdot \boldsymbol{\nabla}_v f = 0.$$

ϕ is the electrostatic potential with $\mathbf{E} = -\nabla\phi$.

Solution:

1. The diamagnetic current is:

$$\mathbf{j}_{dia} = en(\mathbf{u}_{i,dia} - \mathbf{u}_{e,dia}) = -\frac{\nabla p \times \mathbf{B}}{B^2}$$

Using Ampére's law for the magnetic field due to the diamagnetic current (i.e. the θ-component):

$$-\frac{\partial B_{dia}}{\partial r} = \mu_0 j_{dia}$$

Since $\mathbf{B} = \mathbf{B}_0 - \mathbf{B}_{dia}$ and all B-fields just have a z-component, this can be rewritten as:

$$\frac{1}{2}\frac{d(B_0 - B_{dia})^2}{dr} = -\mu_0 \frac{dp}{dr}$$

Integrating and using the boundary condition $B_{dia}(R) = 0$ gives:

$$B(r) = B_0 - B_{dia}(r) = B_0\sqrt{1 - 2\mu_0 p(r)/B_0^2}$$

2a.:

Pressure adds up from ions and electrons: $p = 2n_0T_0(1-(r^2/a^2))^2$ and $dp/dr = -\frac{8}{a^2}n_0T_0r(1-(r^2/a^2))$

Equation of motion:

$$\varrho_{mass}\frac{d\mathbf{u}}{dt} = \varrho_{charge}(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \nabla p \pm \mathbf{R_{e,i}} = 0 \quad \text{(stationary)}$$

multiplying with $\times\mathbf{B}$ gives

$$\varrho(\mathbf{E} \times \mathbf{B} + (\mathbf{u} \times \mathbf{B}) \times \mathbf{B}) - (\nabla p \times \mathbf{B}) \pm \mathbf{R_{e,i}} \times \mathbf{B} = 0$$

Take θ-component and using the fact that

$$(\mathbf{u} \times \mathbf{B}) \times \mathbf{B} = (\mathbf{B} \cdot \nabla) \cdot \mathbf{B} - B^2 \cdot \mathbf{u}$$

where the first term has only a z-component and therefore does not contribute to the perpendicular equation.

$$\varrho(\mathbf{E} \times \mathbf{B} - B^2\mathbf{u}_\perp) - (\nabla p \times \mathbf{B}) \pm \mathbf{R_{e,i}} \times \mathbf{B} = 0$$

Solving for $\mathbf{u}_\perp$

$$\mathbf{u}_\perp = \frac{\mathbf{E} \times \mathbf{B}}{B^2} - \frac{\nabla p \times \mathbf{B}}{\varrho B^2} \pm \frac{\mathbf{R_{e,i}} \times \mathbf{B}}{\varrho B^2}$$

2b:

The first therm is the $E\times B$ drift which is equivalent to the single particle picture; absolute value and direction are independent from charge.

The second term is the diamagnetic drift that does not exist in a single particle description. Different direction for ions and electrons, the resulting current reduces the magnetic field inside the plasma.

The third term is proportional to $u_e - u_i$ and is therefore dominated by the electron contribution.

2c.: $E \times B$:

$$u_\perp = -1000/B[T]m/s$$

diamagnetic term at $r = a/2$:

$$u_\perp = -4T_0/(aqB)100/a[m]B[T]m/s$$

3. Assuming a uniform, i.e. space-independent temperature, only ϕ depends on spatial coordinates. The terms proportional to $\mathbf{E} = -\nabla\phi$ cancel and all other terms of the Vlasov equation are 0. If T is a function of space, the equation

$$\frac{mv^2}{2} + q\phi = \frac{3}{2}kT$$

has to be fulfilled locally. This is the condition for local equlibrium.