High magnetic field limits for collisional plasmas. Gyrokinetic Euler equations

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Numerical methods for the kinetic equations of plasma physics
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Motivations

Energy production through thermonuclear fusion

Magnetic confinement fusion (MCF)

Strongly magnetized plasmas

Landau equation

$$\partial_t f^\varepsilon + v\cdot\nabla_x f^\varepsilon + \frac{q}{m}\left(E(t,x) + v \wedge \frac{B(t,x)}{\varepsilon}\right)\cdot\nabla_v f = Q(f^\varepsilon)$$

f^ε : particle distribution function in (x, v)

$f^\varepsilon(t, x, v)\mathrm{d}x\mathrm{d}v$: particle number inside the volume $\mathrm{d}x\mathrm{d}v$

$Q(f^\varepsilon, f^\varepsilon)$: Landau bilinear collision kernel

Main purposes

- Efficient resolution of problems involving multiple scales
- Landau equation with strong magnetic field is unstable
- Numerical resolution by explicit schemes : small time steps
- Find asymptotic models for different regimes in plasma physics
- Homogenization with respect to the fast cyclotronic motion

Main difficulties

- Complete explicit averaged kernels
- Preserve all the balances (mass, momentum, kinetic energy, entropy)
- Xu & Rosenbluth (linearized around equilibria, implementation seems hard), Garbet (variational principles), Brizard & Hahm
- Explain the perpendicular diffusion in space by averaging
- H theorem, equilibrium, long time behavior, etc

Finite Larmor radius

$$B^\varepsilon = \left(0, 0, \frac{B}{\varepsilon}\right), \quad B > 0$$

$$\omega_c^\varepsilon = \frac{qB^\varepsilon}{m}, \;\; T_{\text{obs}}\, \omega_c^\varepsilon \approx \frac{1}{\varepsilon} >> 1$$

$$\rho_{Larmor} = \frac{|v_\perp|}{\omega_c^\varepsilon} \approx \varepsilon, \quad \frac{L_\perp}{\rho_{Larmor}} \approx 1, \quad \frac{L_\parallel}{\rho_{Larmor}} \approx \frac{1}{\varepsilon} >> 1$$

Landau equation

Notations : $\overline{x} = (x_1, x_2), \; \overline{v} = (v_1, v_2), \; {}^\perp\overline{v} = (v_2, -v_1), \; \omega_c = qB/m$

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}\overline{v}\cdot\nabla_{\overline{x}} f^\varepsilon + v_3 \partial_{x_3} f^\varepsilon + \frac{q}{m} E\cdot\nabla_v f^\varepsilon + \frac{\omega_c}{\varepsilon}(v_2 \partial_{v_1} f^\varepsilon - v_1 \partial_{v_2} f^\varepsilon) = Q(f^\varepsilon, f^\varepsilon)$$

Fast cyclotronic motion

$$\mathcal{T} = b \cdot \nabla_{x,v} = \overline{v} \cdot \nabla_{\overline{x}} f + \omega_c {}^{\perp}\overline{v} \cdot \nabla_{\overline{v}} f$$

$$a \cdot \nabla_{x,v} = v_3 \partial_{x_3} + \frac{q}{m} E \cdot \nabla_v$$

$$\partial_t f^\varepsilon + a \cdot \nabla_{x,v} f^\varepsilon + \frac{1}{\varepsilon} b \cdot \nabla_{x,v} f^\varepsilon = Q(f^\varepsilon, f^\varepsilon)$$

Slow and fast time variables

t : slow time variable, $s = t/\varepsilon$: fast time variable

Particle trajectories

$$(X^\varepsilon(t), V^\varepsilon(t)) = Y^\varepsilon(t) = Y(t, t/\varepsilon) + \varepsilon Y^1(t, t/\varepsilon) + \ldots$$

$$\frac{dY^\varepsilon}{\mathrm{d}t} = a(Y^\varepsilon) + \frac{1}{\varepsilon} b(Y^\varepsilon) \implies \partial_s Y = b(Y)$$

Ansatz

$$f^{\varepsilon} = f + \varepsilon f^1 + \varepsilon^2 f^2 ...$$

$$\mathcal{T} f := b \cdot \nabla_{x,v} f = \overline{v} \cdot \nabla_{\overline{x}} f + \omega_c \, {}^{\perp}\overline{v} \cdot \nabla_{\overline{v}} f = 0$$

$$\partial_t f + v_3 \partial_{x_3} f + \frac{q}{m} E \cdot \nabla_v f + \mathcal{T} f^1 = Q(f,f)$$

Goal : close the evolution equation for f; eliminate the multiplier f^1 thanks to the divergence constraint

Expected limit model

$$\partial_t f + A \cdot \nabla_{x,v} f = \tilde{Q}(f,f), \;\; \mathcal{T} f = 0$$

The constraint

$$b \cdot \nabla_{x,v} f = \mathrm{div}_{x,v}\{fb\} = 0 \leftrightarrow \frac{\mathrm{d}}{\mathrm{d}s}\{f(X(s), V(s))\} = 0$$

Flow of b

$$\frac{\mathrm{d}\overline{X}}{\mathrm{d}s} = \overline{V}(s), \;\; \frac{\mathrm{d}X_3}{\mathrm{d}s} = 0, \;\; \frac{\mathrm{d}\overline{V}}{\mathrm{d}s} = \omega_c \,{}^\perp\overline{V}(s), \;\; \frac{\mathrm{d}V_3}{\mathrm{d}s} = 0$$

Invariants

$$x_1 + \frac{v_2}{\omega_c}, \; x_2 - \frac{v_1}{\omega_c}, \; x_3, \; r = |\overline{v}|, \; v_3$$

$$b\cdot\nabla_{x,v} f = 0 \leftrightarrow \exists \, g : f(t,x,v) = g\left(t, x_1 + \frac{v_2}{\omega_c}, x_2 - \frac{v_1}{\omega_c}, x_3, r = |\overline{v}|, v_3\right)$$

Closure

$$\mathrm{Range}(b \cdot \nabla_{x,v}) \perp \ker(b \cdot \nabla_{x,v})$$

$$P = \mathrm{Proj}_{\ker(b\cdot\nabla_{x,v})} \implies P(\mathrm{Range}(b \cdot \nabla_{x,v})) = 0$$

$$\partial_t f + v_3 \partial_{x_3} f + \frac{q}{m} E \cdot \nabla_v f + \mathcal{T} f^1 = Q(f,f)$$

$$\partial_t f + P(v_3 \partial_{x_3} f + \frac{q}{m} E \cdot \nabla_v f) = P(Q(f,f))$$

How to compute P on transport and collision operators ?

Average along a flow

$$\overline{V}(s) = R(-\omega_c s)\overline{v},\ \ \overline{X}(s) = \overline{x} + \frac{{}^\perp\overline{v}}{\omega_c} - \frac{{}^\perp\overline{V}(s)}{\omega_c},\ \ X_3(s) = x_3,\ \ V_3(s) = v_3$$

Definition (average operator)

$$\langle u \rangle (x,v) = \frac{1}{T_c}\int_0^{T_c} u(X(s;x,v), V(s;x,v))\,\mathrm{d}s \in \ker b\cdot\nabla_{x,v}$$

Proposition The average operator is linear continuous. Moreover it coincides with the orthogonal projection on the kernel of $\mathcal{T}$ *i.e.*,

$$\langle u \rangle \in \ker \mathcal{T}\ :\ \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (u - \langle u \rangle)\varphi\,\mathrm{d}v\mathrm{d}x = 0,\ \ \forall\,\varphi \in \ker \mathcal{T}.$$

Average and first order differential operators

$$\langle a \cdot \nabla_{x,v} f \rangle = \langle \mathrm{div}_{x,v}\{fa\}\rangle = ... = \mathrm{div}_{x,v}\{\langle f \rangle A\} = A \cdot \nabla_{x,v} f$$

Change of coordinates

$$\psi_1 = x_1 + \frac{v_2}{\omega_c}, \;\; \psi_2 = x_2 - \frac{v_1}{\omega_c}, \;\; \psi_3 = x_3, \;\; \psi_4 = \sqrt{(v_1)^2 + (v_2)^2}, \;\; \psi_5 = v_3$$

$$\psi_0 = -\frac{\alpha}{\omega_c}, \;\; \overline{v} = |\overline{v}|e^{i\alpha} = |\overline{v}|(\cos\alpha, \sin\alpha), \;\; \mathcal{T}\psi_0 = 1$$

$$u(x,v) = U(\psi_0(x,v), \psi_1(x,v), ..., \psi_5(x,v))$$

Derivations along the invariants

$$b^i \cdot \nabla_{x,v} u = \frac{\partial U}{\partial \psi_i}(\psi(x,v)), \;\; 0 \le i \le 5$$

Expressions for b^i

$$b^0 \cdot \nabla_{x,v} = \overline{v} \cdot \nabla_{\overline{x}} + \omega_c \, {}^\perp\overline{v} \cdot \nabla_{\overline{v}}, \ldots, b^4 \cdot \nabla_{x,v} = -\frac{{}^\perp\overline{v}}{\omega_c |\overline{v}|} \cdot \nabla_{\overline{x}} + \frac{\overline{v}}{|\overline{v}|} \cdot \nabla_{\overline{v}}$$

Remark

$$[b^i, b^j] = 0, \;\; 0 \le i, j \le 5.$$

Proposition Assume that $[c, b] = 0$. Then the operator $\mathrm{div}_{x,v}(\cdot \; c)$ is commuting with the average operator associated to the flow of $b \cdot \nabla_{x,v}$ (derivation w.r.t. a parameter under the integral sign)

$$\mathrm{div}_{x,v}(\langle u \rangle \, c) = \langle \mathrm{div}_{x,v}(uc) \rangle \,, \;\; c \cdot \nabla_{x,v} \langle u \rangle = \langle c \cdot \nabla_{x,v} u \rangle \,.$$

Proof

$$[c, b] = 0 \leftrightarrow Z(h; Y(s; y)) = Y(s; Z(h; y))$$

How average and divergence commute

$$\xi = \sum_i (\xi \cdot \nabla_{x,v}\psi_i) b^i$$

$$\langle \mathrm{div}_{x,v}\xi \rangle = \left\langle \sum_{i=0}^{5} \mathrm{div}_{x,v}\{(\xi \cdot \nabla_{x,v}\psi_i) b^i\} \right\rangle = \mathrm{div}_{x,v} \left\{ \sum_{i=0}^{5} \langle \xi \cdot \nabla_{x,v}\psi_i \rangle \, b^i \right\}$$

$$\langle a \cdot \nabla_{x,v} f \rangle = ?, \quad a \cdot \nabla_{x,v} = v_3 \partial_{x_3} + \frac{q}{m} E \cdot \nabla_v$$

$$\left\langle v_3 \partial_{x_3} f + \frac{q}{m} E \cdot \nabla_v f \right\rangle = \frac{\langle {}^\perp \overline{E} \rangle}{B} \cdot \nabla_{\overline{x}} f + v_3 \partial_{x_3} f + \frac{q}{m} \langle E_3 \rangle \, \partial_{v_3} f$$

How to average the Landau kernel?

Average and collisions

Fokker-Planck-Landau kernel : integral differential operator (second order derivatives and convolution)

Relaxation operator

$$Q_B(f)(x,v) = \frac{1}{\tau} \int_{\mathbb{R}^3} s(v,v')\{M(v)f(x,v') - M(v')f(x,v)\}\,\mathrm{d}v'$$

$$\int_{\mathbb{R}^3} Q_B(f)(v) f(v)\,\frac{\mathrm{d}v}{M} = -\frac{1}{2\tau} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} sMM' \left[\frac{f(v)}{M(v)} - \frac{f(v')}{M(v')}\right]^2 \mathrm{d}v'\,\mathrm{d}v \le 0$$

Proposition

For any function $f \in \ker \mathcal{T}$ we have

$$\left\langle \int_{\mathbb{R}^3} C(v,v') f(x,v')\,\mathrm{d}v' \right\rangle = \omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \mathcal{C}(|\overline{v}|, v_3, |\overline{v'}|, v_3', z) f(\overline{x'}, x_3, v')\mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'$$

where $z = \omega_c \overline{x} + {}^\perp\overline{v} - (\omega_c \overline{x'} + {}^\perp\overline{v'})$.

Corollary Assume that $s(v,v') = \sigma(|v-v'|), \;\; v, v' \in \mathbb{R}^3$. For any $f \in \ker \mathcal{T}$ we have

$$\langle Q_B f \rangle = \frac{\omega_c^2}{\tau} \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \mathcal{S}(|\overline{v}|, v_3, |\overline{v'}|, v_3', z)\{M(v) f(\overline{x'}, x_3, v') - M(v') f(x,v)\}$$

with $z = \omega_c \overline{x} + {}^\perp\overline{v} - (\omega_c \overline{x'} + {}^\perp\overline{v'})$ and

$$\mathcal{S}(r, v_3, r', v_3', z) = \sigma(\sqrt{|z|^2 + (v_3 - v_3')^2}\,)\chi(r, r', z)$$

$$\chi(r, r', z) = \frac{\mathbf{1}_{\{|r-r'|<|z|<r+r'\}}}{\pi^2 \sqrt{|z|^2 - (r-r')^2}\sqrt{(r+r')^2 - |z|^2}}$$

Averaged relaxation operator

1. non local in space
2. similar properties (mass balance, negativity) but globally in $(\overline{x}, v)$
3. averaging leads to convolution with respect to the invariants

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \langle Q_B \rangle (f) \frac{f}{M} \,\mathrm{d}v\mathrm{d}x = -\frac{\omega_c^2}{2\tau} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \mathcal{S}(|\overline{v}|, v_3, |\overline{v'}|, v_3', z) M M'$$
$$\times \left[\frac{f(x,v)}{M(v)} - \frac{f(\overline{x'}, x_3, v')}{M(v')} \right]^2 \mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'\mathrm{d}v\mathrm{d}x \leq 0.$$

The Fokker-Planck kernel

$$\partial_t f + \frac{\left\langle {}^\perp \overline{E} \right\rangle}{B} \cdot \nabla_{\overline{x}} f + v_3 \partial_{x_3} f + \frac{q}{m} \langle E_3 \rangle \partial_{v_3} f = \langle Q_{FP} \rangle (f)$$

$$f(0, x, v) = \left\langle f^{\mathrm{in}} \right\rangle (x, v)$$

$$Q_{FP}(f) = \frac{\theta}{m\tau} \mathrm{div}_v \left(\nabla_v f + \frac{m}{\theta} v f \right) = \frac{\theta}{m\tau} \mathrm{div}_v \left\{ M \nabla_v \left(\frac{f}{M} \right) \right\}$$

$$\langle Q_{FP} \rangle f(x, v) = \frac{\theta}{m\tau} \mathrm{div}_{\omega_c x, v} \left\{ M \mathcal{L} \nabla_{\omega_c x, v} \left(\frac{f}{M} \right) \right\}$$

$$\mathcal{L} = \begin{pmatrix} 2(I_3 - e_3 \otimes e_3) & -E \\ E & I_3 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The Fokker-Planck-Landau kernel

$$Q_{FPL}(f,f)(v) = \mathrm{div}_v \int_{\mathbb{R}^3} \sigma S(v-v') \left(f(v')(\nabla_v f)(v) - f(v)(\nabla_{v'} f)(v') \right) \mathrm{d}v'$$

Mass, momentum, kinetic energy balances

$$\int_{\mathbb{R}^3} Q_{FPL}(f,f)\,\mathrm{d}v = 0, \int_{\mathbb{R}^3} v Q_{FPL}(f,f)\,\mathrm{d}v = 0, \int_{\mathbb{R}^3} \frac{|v|^2}{2} Q_{FPL}(f,f)\,\mathrm{d}v = 0$$

Entropy production

$$D := - \int_{\mathbb{R}^3} \ln f \; Q_{FPL}(f,f)\,\mathrm{d}v \geq 0$$

The gain kernel Q^+_{FPL}

For any function $f = f(x, v)$ satisfying the constraint $\mathcal{T} f = 0$ we have

$$\langle Q^+_{FPL}(f,f)\rangle = \mathrm{div}_{\omega_c x, v}\{\omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma(\sqrt{|z|^2 + (v_3 - v_3')^2}\,) f(x_1', x_2', x_3, v') \times \chi(|\overline{v}|, |\overline{v'}|, z)\, A^+ \nabla_{\omega_c x, v} f(x, v)\, \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2'\}$$

The loss kernel Q^-_{FPL}

For any function $f = f(x, v)$ satisfying the constraint $\mathcal{T} f = 0$ we have

$$\langle Q^-_{FPL}(f,f)\rangle = \mathrm{div}_{\omega_c x, v}\{\omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma(\sqrt{|z|^2 + (v_3 - v_3')^2}\,) f(x, v) \times \chi(|\overline{v}|, |\overline{v'}|, z)\, A^- \nabla_{\omega_c x', v'} f(x_1', x_2', x_3, v')\, \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2'\}$$

$$
\begin{aligned}
&\langle Q_{FPL}(f,f)\rangle (x,v)\\
&= \mathrm{div}_{\omega_c x, v}\left\{\omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma\chi f(\overline{x'}, x_3, v') A^{+} \nabla_{\omega_c x, v} f(x,v)\, \mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'\right\}\\
&- \mathrm{div}_{\omega_c x, v}\left\{\omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma\chi f(x,v) A^{-} \nabla_{\omega_c x', v'} f(\overline{x'}, x_3, v')\, \mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'\right\}
\end{aligned}
$$

and

$$
\sigma\chi A^{+}(r, v_3, r', v_3', z) = \sum_{i=1}^{4} \xi^i(\overline{x}, v, \overline{x'}, v') \otimes \xi^i(\overline{x}, v, \overline{x'}, v')
$$

$$
\sigma\chi A^{-}(r, v_3, r', v_3', z) = \sum_{i=1}^{4} \varepsilon_i \xi^i(\overline{x}, v, \overline{x'}, v') \otimes \xi^i(\overline{x'}, v', \overline{x}, v)
$$

for some vector fields $(\xi^i)_{1\le i\le 4}$ and $\varepsilon_1 = \varepsilon_2 = -1, \varepsilon_3 = \varepsilon_4 = 1$

$$\xi^1 = \{\sigma\chi\}^{1/2} \frac{r' \sin\varphi\,(v_3 - v_3')}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}} \left(\frac{(\overline{v}, 0)}{|\overline{v}|}, \frac{(\,^\perp\overline{v}, 0)}{|\overline{v}|} \right)$$

$$\xi^2 = \{\sigma\chi\}^{1/2} \left[\frac{r - r' \cos\varphi}{|z|} \left(\frac{(\overline{v}, 0)}{|\overline{v}|}, \frac{(\,^\perp\overline{v}, 0)}{|\overline{v}|} \right) + \left(\frac{(^\perp z, 0)}{|z|}, 0 \right) \right]$$

$$\xi^3 = \{\sigma\chi\}^{1/2} \frac{r' \sin\varphi}{|z|} \left(\frac{(\,^\perp\overline{v}, 0)}{|\overline{v}|}, -\frac{(\overline{v}, 0)}{|\overline{v}|} \right)$$

$$\frac{\xi^4}{\{\sigma\chi\}^{1/2}} = \frac{(r' \cos\varphi - r)(v_3 - v_3')}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}} \left(\frac{(\,^\perp\overline{v}, 0)}{|\overline{v}|}, -\frac{(\overline{v}, 0)}{|\overline{v}|} \right) + \frac{\left((v_3 - v_3') \frac{(z,0)}{|z|}, -|z| e_3 \right)}{\sqrt{|z|^2 + (v_3 - v_3')^2}}$$

Averaged Fokker-Planck-Landau kernel

1. non local in space
2. averaging leads to diffusion both in perpendicular space directions and velocity and convolution with respect to the invariants
3. similar properties (mass/momentum/kinetic energy balances, entropy decreasing) but globally in $(\overline{x}, v)$

Theorem H Consider two functions $f = f(x, v), \varphi = \varphi(x, v)$. We have

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \langle Q_{FPL} \rangle (f, f)\varphi \,\mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 = -\frac{\omega_c^2}{2}\sum_{i=1}^{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} ff'$$

$$(\xi^i \cdot \nabla \ln f - \varepsilon_i(\xi^i)'\nabla' \ln f')(\xi^i \cdot \nabla\varphi - \varepsilon_i(\xi^i)'\nabla'\varphi')\,\mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'\,\mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2$$

where

$$f = f(x, v),\ f' = f'(x_1', x_2', x_3, v')$$

$$\nabla\varphi = \nabla_{\omega_c x, v}\varphi(x, v),\ \nabla'\varphi' = \nabla_{\omega_c x', v'}\varphi(x_1', x_2', x_3, v')$$

$$\xi^i = \xi^i(x_1, x_2, v, x_1', x_2', v'),\ \ (\xi^i)' = \xi^i(x_1', x_2', v', x_1, x_2, v).$$

In particular the entropy $f \ln f$ (globally in (x_1, x_2, v)) decreases

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \ln f \,\langle Q_{FPL} \rangle (f, f)\,\mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 \le 0.$$

Average collision invariants

$$\xi^i \cdot \nabla \varphi - \varepsilon_i (\xi^i)' \cdot \nabla' \varphi' = 0, \ \forall i \Leftrightarrow \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \langle Q_{FPL} \rangle (f,f) \varphi \, \mathrm{d}v \mathrm{d}x_1 \mathrm{d}x_2 = 0 \ \forall f$$

$$1, \ \omega_c \overline{x} + {}^{\perp}\overline{v}, \ v_3, \ \frac{|v|^2}{2}, \ \frac{|\omega_c \overline{x} + {}^{\perp}\overline{v}|^2 - |\overline{v}|^2}{2}$$

Gyro-kinetic equilibria

$$\xi^i \cdot \nabla \ln f - \varepsilon_i (\xi^i)' \cdot \nabla' \ln f' = 0, \ \forall i \Leftrightarrow \langle Q_{FPL} \rangle (f,f) = 0$$

Product of $(\overline{x}, v)$ Maxwellians

$$f \sim \frac{1}{2\pi \frac{\mu\theta}{\mu - \theta}} \exp\left(- \frac{|\overline{v}|^2}{2 \frac{\mu\theta}{\mu-\theta}} \right) \frac{1}{(2\pi\theta)^{1/2}} \exp\left(- \frac{(v_3 - u_3)^2}{2\theta} \right)$$
$$\times \frac{1}{2\pi\mu} \exp\left(- \frac{|\omega_c \overline{x} + {}^{\perp}\overline{v} - \overline{u}|^2}{2\mu} \right).$$

$$\mathcal{E} = \frac{\rho}{(2\pi)^{5/2} \frac{\mu^2 \theta^{3/2}}{\mu - \theta}} \exp\left(- \frac{|\overline{v}|^2 + (v_3 - u_3)^2}{2\theta} - \frac{|\omega_c \overline{x} + {}^{\perp}\overline{v} - \overline{u}|^2 - |\overline{v}|^2}{2\mu} \right)$$

Linearization around equilibria

$$\langle Q_{FPL}\rangle(f,f) = \langle Q_{FPL}\rangle(f,f) - \langle Q_{FPL}\rangle(\mathcal{E}_f,\mathcal{E}_f)$$
$$\approx \langle Q_{FPL}\rangle(\mathcal{E}_f, f-\mathcal{E}_f) + \langle Q_{FPL}\rangle(f-\mathcal{E}_f,\mathcal{E}_f) := \mathcal{L}(f)$$

Theorem H Consider two functions $f = f(x,v), \varphi = \varphi(x,v)$. We have

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\mathcal{L}(f)\varphi \,\mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 = -\frac{\omega_c^2}{2}\sum_{i=1}^4\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\mathcal{E}_f\mathcal{E}_f'$$
$$(\xi^i\cdot\nabla\frac{f}{\mathcal{E}_f} - \varepsilon_i(\xi^i)'\nabla'\frac{f'}{\mathcal{E}_f'})(\xi^i\cdot\nabla\varphi - \varepsilon_i(\xi^i)'\nabla'\varphi')\,\mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'\,\mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2$$

Collisional invariants

$$\xi^i\cdot\nabla\varphi - \varepsilon_i(\xi^i)'\cdot\nabla'\varphi' = 0,\ \forall i \Leftrightarrow \int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\mathcal{L}(f)\varphi\,\mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 = 0\ \forall f$$

Negativity

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\frac{f}{\mathcal{E}_f}\mathcal{L}(f)\,\mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 \le 0$$

Equilibria parametrization

For any $(\rho, u_1, u_2, u_3, K, G) \in \mathbb{R}^6$, $\rho > 0, K > 0, K + G > 0$ there is a unique local (in x_3) equilibrium $f = f(\overline{x}, v)$ for $\langle Q_{FPL} \rangle$ satisfying

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} f \,\mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 = \rho, \quad \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} (\omega_c \overline{x} + {}^{\perp}\overline{v}, v_3) f \,\mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 = \rho u$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|v|^2}{2} f \,\mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 = \rho \frac{(u_3)^2}{2} + \rho K$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\omega_c \overline{x} + {}^{\perp}\overline{v}|^2 - |\overline{v}|^2}{2} f \,\mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 = \rho \frac{|\overline{u}|^2}{2} + \rho G$$

$$\frac{\mu\theta}{\mu - \theta} + \frac{\theta}{2} = K, \quad \mu - \frac{\mu\theta}{\mu - \theta} = G$$

Fluid models around equilibria

$$\partial_t f^\tau + v_3 \partial_{x_3} f^\tau + \frac{q}{m} E_3(t, x_3) \partial_{v_3} f^\tau = \frac{1}{\tau} \langle Q_{FPL} \rangle (f^\tau, f^\tau)$$

$$f^\tau = f + \tau f^1 + \tau^2 f^2 + \dots$$

$$\langle Q_{FPL} \rangle (f, f) = 0 \Leftrightarrow f = \mathcal{E}_{\rho, u, \theta, \mu}$$

Collision invariants

$$\varphi \in \left\{ 1, \;\; \omega_c \overline{x} + {}^\perp \overline{v}, \;\; v_3, \;\; \frac{|v|^2}{2}, \;\; \frac{|\omega_c \overline{x} + {}^\perp \overline{v}|^2 - |\overline{v}|^2}{2} \right\}$$

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \left\{ \partial_t f + v_3 \partial_{x_3} f + \frac{q}{m} E_3(t, x_3) \partial_{v_3} f \right\} \varphi \, \mathrm{d}v \mathrm{d}x_1 \mathrm{d}x_2 = 0$$

Gyrokinetic Euler equations

$$\partial_t \rho + \partial_{x_3}(\rho u_3) = 0$$

$$\partial_t(\rho u) + \partial_{x_3}(\rho(u_3 u + (0,0,\theta))) - \rho\frac{q}{m}(0,0,E_3) = 0$$

$$\partial_t\left[\rho\left(\frac{\mu\theta}{\mu-\theta} + \frac{\theta + u_3^2}{2}\right)\right] + \partial_{x_3}\left[u_3\rho\left(\frac{\mu\theta}{\mu-\theta} + \frac{3\theta + u_3^2}{2}\right)\right] - \frac{q}{m}E_3\rho u_3 = 0$$

$$\partial_t\left[\rho\left(\mu - \frac{\mu\theta}{\mu-\theta}\right)\right] + \partial_{x_3}\left[\rho u_3\left(\mu - \frac{\mu\theta}{\mu-\theta}\right)\right] = 0$$

Entropy

$$\partial_t\left(\rho \ln \frac{\rho(\mu-\theta)}{\mu^2\theta^{3/2}}\right) + \partial_{x_3}\left(\rho u_3 \ln \frac{\rho(\mu-\theta)}{\mu^2\theta^{3/2}}\right) = 0$$

Conclusions

- exact computations of the averaged collison kernels

- mass, momentum, kinetic energy balances

- H theorem

- complete description of the gyrokinetic equilibria and collision invariants

- Euler equations, Navier-Stokes equations

Perspectives

- numerical simulation by well adaptated schemes

- macro-micro decomposition (implicite scheme for the zero average part, explicite scheme for the average part)

- general magnetic shape

- coupling with Maxwell equations