Asymptotic preserving schemes for the long time behavior of Vlasov equations with a strong external magnetic field

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Motivation

Construct numerical schemes for the long time behavior of Vlasov equations of the form

$$\partial_t f + v\cdot\nabla_x f + \frac{q}{m}(E + v\times B)\cdot\nabla_v f = 0, \qquad f(t=0,x,v) = f_0(x,v).$$

The unknown is $f(t,x,v)$, $t \geq 0$, $(x,v) \in \mathbb{R}^2 \times \mathbb{R}^2$.

We are interested in the long time behavior of strongly magnetized plasmas such that, after suitable scaling, we have

$$\partial_t f + \frac{1}{\varepsilon_t} v \cdot \nabla_x f + \frac{1}{\varepsilon_t}\left(E + v \times \frac{B}{\varepsilon_b}\right) \cdot \nabla_v f = 0.$$

The limit has been studied theoretically in Golse-Saint-Raymond 99, Frénod-Sonnendrücker 00, Bostan 10.

Aim: construct a Asymptotic Preserving scheme which

- is free from the constraint $\Delta t = \mathcal{O}(\varepsilon_t \varepsilon_b)$ (stability).
- is consistent with all regimes, $\varepsilon = \mathcal{O}(1)$ AND $\varepsilon \ll 1$.

Principle of AP schemes [1]

Let P_ε a continuous problem that converges to P_0 as $\varepsilon \to 0$.
We seek an approximation $P_{\varepsilon,h}$ such that

- for fixed $\varepsilon > 0$, $P_{\varepsilon,h}$ is a consistent and stable discretization of P_ε
- as $\varepsilon \to 0$, $P_{\varepsilon,h}$ converges to a consistent and stable discretization $P_{0,h}$ of P_0.

$$
\begin{array}{ccc}
P_{\varepsilon,h} & \xrightarrow{h\to 0} & P_\varepsilon \\
{\scriptstyle \varepsilon\to 0}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon\to 0} \\
P_{0,h} & \xrightarrow[h\to 0]{} & P_0
\end{array}
$$

[1] S. Jin, SIAM J. Sci. Comput. 99

Such a scheme is well suited for multiscale problems where the asymptotic (or averaged) model is well identified.

- No coupling between models (for instance when $\varepsilon = \varepsilon(t)$).
- Discretization of the microscopic model with constant cost with respect to ε.
- Enables to capture the limit regime ("Asymptotic Preserving" property).

"Asymptotic Preserving" schemes are now well developed in collisional kinetic theory with applications to plasmas, rarefied gas dynamics, radiative transfer ...

see Klar 99, Jin-Pareschi-Toscani 00, Jin 01, Lemou-Mieussens 08, Carrillo-Goudon-Lafitte-Vecil 08, Filbet-Jin 10, Dimarco-Pareschi 11, ...

Outline of the talk

In the sequel, we restrict to the transverse 4 dimensional case in a slab geometry: $B = (0, 0, 1)$ such that $v \times B = v^{\perp} = (v_2, -v_1, 0)$. The initial model then reduces to

$$\partial_t f + \frac{1}{\varepsilon} v \cdot \nabla_x f + \frac{1}{\varepsilon}\left(E + \frac{1}{\varepsilon} v^{\perp}\right) \cdot \nabla_v f = 0$$

where

- $f(t, x_1, x_2, v_1, v_2)$ is the distribution function,
- $E(t, x_1, x_2)$ satisfies $\nabla_x \cdot E = \int_{\mathbb{R}^2} f(t, x_1, x_2, v_1, v_2) dv_1 dv_2 - 1$,
- $x_1, x_2 \in [0, L_1] \times [0, L_2]$

1. $Tf = v \cdot \nabla_x f + E \cdot \nabla_v f$
2. $Lf = v^{\perp} \cdot \nabla_v f$

so that the model rewrites

$$\partial_t f + \frac{1}{\varepsilon} Tf + \frac{1}{\varepsilon^2} Lf = 0.$$

Polar coordinates in $(v_1, v_2) \longrightarrow (r, \theta)$ with $r \geq 0$ and $\theta \in \mathbb{T} = [0, 2\pi]$ so that

$$\begin{aligned} Tf &= r\cos\theta\partial_{x_1} f + r\sin\theta\partial_{x_2} f + \\ &\quad (E_1\cos\theta + E_2\sin\theta)\partial_r f + \frac{1}{r}(-E_1\cos\theta + E_2\sin\theta)\partial_\theta f \end{aligned}$$

and

$$Lf = -\partial_\theta f$$

Leads to more complicated terms... but L is very simple...

Asymptotic model

$$\partial_t f + \frac{1}{\varepsilon} Tf + \frac{1}{\varepsilon^2} Lf = 0.$$

As $\varepsilon \to 0$, the sequence $f = f^\varepsilon$ converges formally to f^0 which belongs to the kernel of $L = -\partial_\theta$. So that we need some useful properties

- L is skew-symmetric for the L^2 scalar product on $\mathbb{T}$
- the kernel of L is the set of functions independent of $\theta \in \mathbb{T}$
- the projector on the kernel of L is the average operator

$$\Pi h = \frac{1}{2\pi} \int_{\mathbb{T}} h(\theta) d\theta.$$

- L is invertible on the orthogonal on its kernel:

if $\Pi h = 0$ then $L^{-1} h = (\mathrm{I} - \Pi) \int_0^\tau h(\theta') d\theta'$

It is convenient to decompose the unknown as

$$f(t, x_1, x_2, r, \theta) = G(t, x_1, x_2, r) + h(t, x_1, x_2, r, \theta),$$

with

$$G = \Pi f \ \text{(average in } \theta), \qquad h = (\mathrm{I} - \Pi) f.$$

These quantities (G, h) satisfy the micro-macro system

$$\begin{cases} \partial_t G + \dfrac{1}{\varepsilon} \Pi T(G + h) = 0 \\ \partial_t h + \dfrac{1}{\varepsilon} (\mathrm{I} - \Pi) T(G + h) = -\dfrac{1}{\varepsilon^2} L h \end{cases}$$

Since $\Pi h = 0$, one can invert the second equation:

$$\begin{aligned} h &= -\varepsilon L^{-1} \left[\partial_t h + (\mathrm{I} - \Pi) T(G + h) \right] \\ &= -\varepsilon L^{-1} \left[(\mathrm{I} - \Pi) T G \right] + \mathcal{O}(\varepsilon^2) \end{aligned}$$

Inserting in the macro equation yields to

$$\partial_t G + \frac{1}{\varepsilon}\Pi T(G - \varepsilon L^{-1}[(\mathrm{I} - \Pi)TG]) = \mathcal{O}(\varepsilon)$$

But $\Pi TG = 0$ and $\Pi T(L^{-1}[(\mathrm{I} - \Pi)TG]) = -E^{\perp} \cdot \nabla_x G$ so that the asymptotic model is the guiding-center model (or 2D Euler)

$$\partial_t G + E^{\perp} \cdot \nabla_x G = 0, \quad \nabla_x \cdot E(x_1, x_2) = \int_0^{+\infty} G(x_1, x_2, r) r dr - 1.$$

- with a general magnetic field, we recover the $E \times B/|B^2|$ term,
- considering the full $6D$ problem, we recover the drift-kinetic model.

Time discretization: $G(t^n) \longleftrightarrow G^n$ and $h(t^n) \longleftrightarrow h^n$

- first order in space (upwind scheme $\Psi(F^n) := \Psi_{i,j,k,\ell}(F^n) \approx (TF)(t^n, x_{i,j}, r_k, \theta_\ell)$)
- Fourier in θ for $L = -\partial_\theta$ (invertible on null averaged functions)

$$\begin{cases} h^{n+1} & = h^n - \frac{\Delta t}{\varepsilon}(I - \Pi)\Psi(G^n + h^n) - \frac{\Delta t}{\varepsilon^2} L h^{n+1} \\ G^{n+1} & = G^n - \frac{\Delta t}{\varepsilon}\Pi\Psi(h^{n+1}) \end{cases}$$

AP property: when $\varepsilon \ll 1$, we have from the first equation

$$h^{n+1} = (I + \frac{\Delta t}{\varepsilon^2} L)^{-1}[h^n - \frac{\Delta t}{\varepsilon}(I - \Pi)\Psi(G^n + h^n)] = -\varepsilon L^{-1}(I - \Pi)\Psi(G^n) + \mathcal{O}(\varepsilon^2),$$

which injected in the macro equation, yields

$$\begin{aligned} G^{n+1} &= G^n - \frac{\Delta t}{\varepsilon}\Pi\Psi\left(-\varepsilon L^{-1}(I - \Pi)\Psi(G^n)\right) + \mathcal{O}(\varepsilon) \\ &= G^n - \Delta t\,\Pi\Psi\left(L^{-1}(I - \Pi)\Psi(G^n)\right) + \mathcal{O}(\varepsilon) \end{aligned}$$

Improvement

From

$$\begin{cases} h^{n+1} &= h^n - \frac{\Delta t}{\varepsilon}(I-\Pi)T(G^n + h^n) - \frac{\Delta t}{\varepsilon^2} L h^{n+1} \\ G^{n+1} &= G^n - \frac{\Delta t}{\varepsilon}\Pi T(h^{n+1}) \end{cases}$$

we have

$$\begin{aligned} h^{n+1} &= \left(\mathrm{I} + \frac{\Delta t}{\varepsilon^2} L\right)^{-1} \left(h^n - \frac{\Delta t}{\varepsilon}(I-\Pi)T(G^n + h^n)\right) \\ &= \left(\mathrm{I} + \frac{\Delta t}{\varepsilon^2} L\right)^{-1} \left(-\frac{\Delta t}{\varepsilon} T(G^n)\right) + \varepsilon^2 \mathcal{F}(h^n) \\ &= -\varepsilon \mathcal{B}^\varepsilon \cdot \nabla_{x,v} G^n + \varepsilon^2 \mathcal{F}(h^n) \end{aligned}$$

which, injected in the macro equation, gives

$$
\begin{aligned}
G^{n+1} &= G^n - \frac{\Delta t}{\varepsilon}\Pi T(h^{n+1}) \\
&= G^n - \frac{\Delta t}{\varepsilon}\Pi T(-\varepsilon\mathcal{B}^\varepsilon\cdot\nabla G^n) - \frac{\Delta t}{\varepsilon}\Pi T(\varepsilon^2\mathcal{F}(h^n)) \\
&= G^n - \frac{\Delta t}{\varepsilon^4/\Delta t^2+1}E^\perp\cdot\nabla_x G^n + \frac{\varepsilon^2\Delta t^2}{\varepsilon^4+\Delta t^2}\mathcal{D}G^n - \varepsilon\Delta t\Pi T(\mathcal{F}(h^n))
\end{aligned}
$$

Then, from (G^n, h^n), the algorithm becomes

- compute G^{n+1}
- compute $h^{n+1} = h^n - \frac{\Delta t}{\varepsilon}(I-\Pi)T(G^n+h^n) - \frac{\Delta t}{\varepsilon^2}Lh^{n+1}$

As $\varepsilon\to 0$, we have easily $h^{n+1}\to 0$ and

$$G^{n+1} = G^n - \Delta t\, E^\perp\cdot\nabla_x G^n,$$

which is a consistent scheme of the asymptotic model (guiding-center equation).

Numerical scheme

Extension to second order (in phase space)
Modified simple Lax-Ritchmyer numerical scheme:

- two steps in time (prediction-correction scheme)
- second order in space (centered finite difference $\Phi(F^n) := \Phi_{i,j,k,\ell}(F^n) \approx TF(t^n, x_{i,j}, r_k, \theta_\ell)$)
- $\widetilde{F}^n_{i,j,k,\ell} = \left(F^n_{i\pm1,j,k,\ell} + F^n_{i,j\pm1} + F^n_{i,j,k\pm1,\ell} + F^n_{i,j,k,\ell\pm1} \right) /8$ is an average to stabilize the centered scheme (in the prediction step)
- Fourier in θ for the inversion of L

Prediction step

$$\begin{cases} h^{n+1/2} = \widetilde{h}^n - \frac{\Delta t}{2\varepsilon}(I-\Pi)\Phi(G^n + h^n) - \frac{\Delta t}{2\varepsilon^2} L h^{n+1/2}, \\ G^{n+1/2} = \widetilde{G}^n - \frac{\Delta t}{2\varepsilon}\Pi\Phi h^{n+1/2}, \end{cases}$$

Correction step

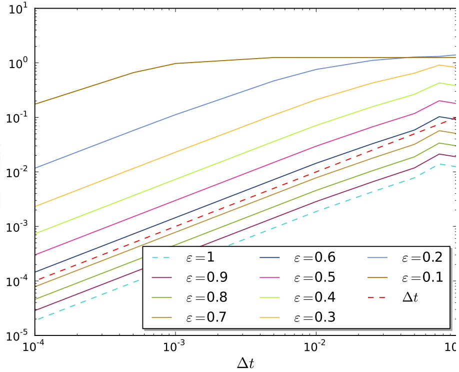
$$\begin{cases} G^{n+1} = G^n - \frac{\Delta t}{\varepsilon}\Pi\Phi(G^{n+1/2} + h^{n+1/2}). \\ h^{n+1} = h^n - \frac{\Delta t}{\varepsilon}(I-\Pi)\Phi(G^{n+1/2} + h^{n+1/2}) - \frac{\Delta t}{\varepsilon^2} L h^{n+1} \end{cases}$$

The same improvements as before can be performed

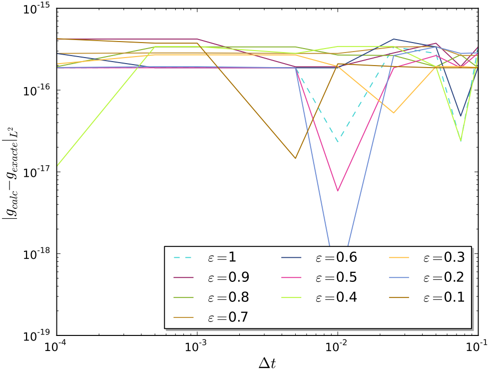
Numerical results (I)

We first consider $E = 0$ and the initial condition $f_0(x_1, x_2, v_1, v_2) = \frac{1}{2\pi} e^{-|v|^2/2}(1 + v_1)$ so that the solution writes

$$f(t, x_1, x_2, v_1, v_2) = \frac{1}{2\pi} e^{-|v|^2/2}(1 + \cos(t/\varepsilon^2)v_1 - \sin(t/\varepsilon^2)v_2)$$



error on f as a function of Δt



error on G as a function of Δt

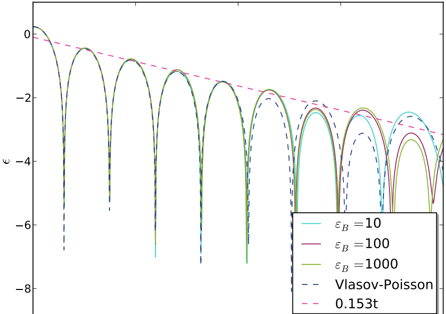
Validation on Vlasov-Poisson test cases

Considering $\varepsilon_t = 1$, we have

$$\partial_t f + v \cdot \nabla_x f + \left(E + \frac{B}{\varepsilon_b} v^{\perp} \right) \cdot \nabla_v f = 0,$$

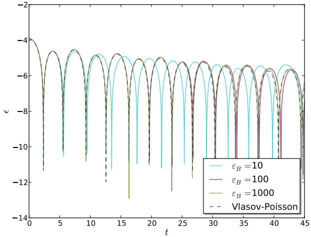
so that considering $\varepsilon_b \gg 1$, one has to recover $4D$ Vlasov-Poisson results.

We first consider E throughout Poisson and the initial condition $f_0(x_1, x_2, v_1, v_2) = \frac{1}{2\pi} e^{-|v|^2/2}(1 + 0.01 \cos(kx_1))$

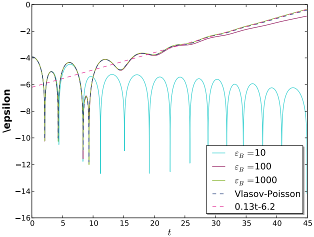


Validation on Vlasov-Poisson test cases

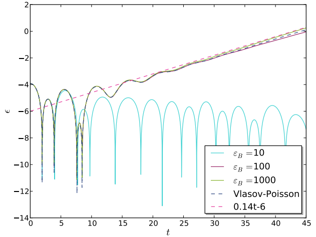
We first consider E throughout Poisson and the initial condition
$$f_0(x_1,x_2,v_1,v_2) = \frac{1}{4\pi}\left(e^{-|v-v_0|^2/2} + e^{-|v+v_0|^2/2}\right)(1+0.01\cos(kx_1))$$



$v_0 = 1.3$



$v_0 = 2.4$



$v_0 = 3$

VALIDATION ON THE FULL MODEL: $\varepsilon \to 0$

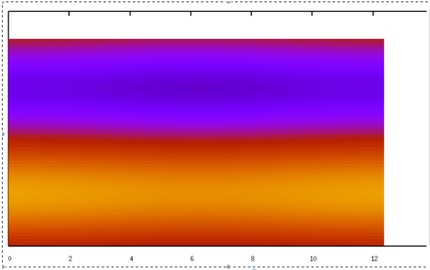
We first consider E throughout Poisson and the initial condition

$$f_0(x_1, x_2, v_1, v_2) = \frac{1}{2\pi} e^{-|v|^2/2}(1 + \sin(x_2) + 0.05\cos(kx_1)),$$

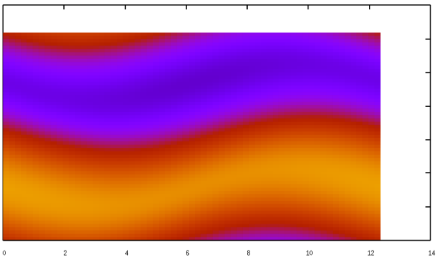
with $x_1 \in [0, 2\pi/k], x_2 \in [0, 2\pi], r \in [0, 5], \theta \in [0, 2\pi]$.

We consider the time evolution of the Fourier mode $(1, 1)$ of the electric potential $\phi(x_1, x_2)$, as suggested in Shoucri (IJNME 79).

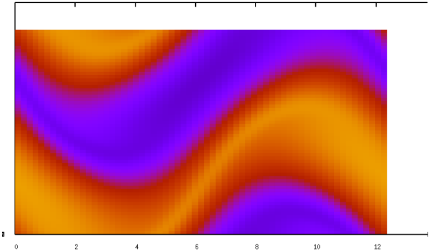
The numerical parameters are $N_x = N_y = 32, N_r = N_\theta = 16$, $\Delta t = 0.01$.



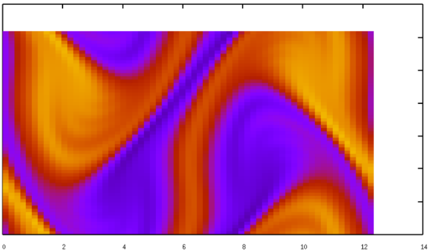
$t = 0$



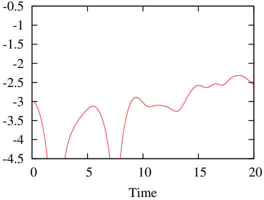
$t = 10$



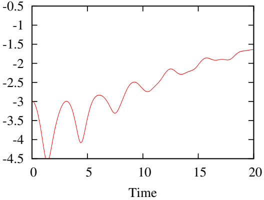
$t = 20$



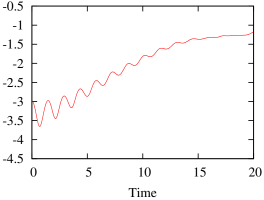
$t = 30$



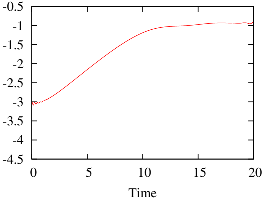
$\varepsilon = 1$



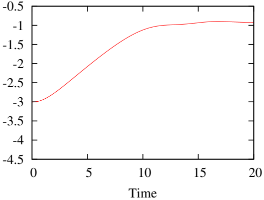
$\varepsilon = 0.75$



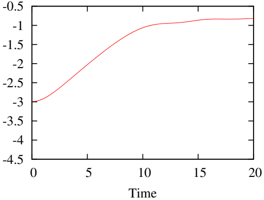
$\varepsilon = 0.5$



$\varepsilon = 0.2$

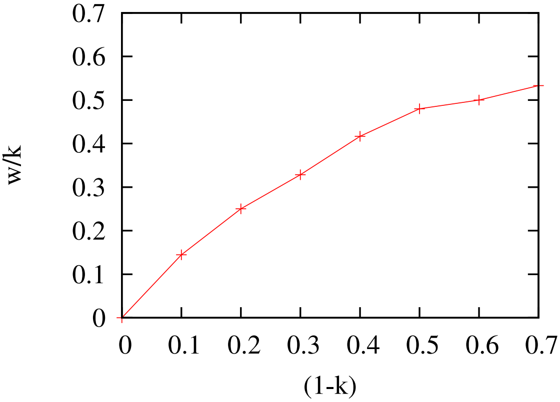


$\varepsilon = 0.01$



ε" = "0

The instability rate can be computed for ε small enough[2]



[2]Shoucri (IJNME 79), C.-Mehrenberger-Sonnendrücker (JCP 10)

Conclusions/Perspectives

- AP scheme in the strong magnetic field limit
- First academic validation

Possible improvements

- numerical scheme in phase space (to avoid spurious oscillations)
- non homogeneous magnetic field $B = B(x, y)$

Perspectives

- other gyrokinetic type limits (Finite Larmor radius) ?
- capturing the oscillations instead of the average only (uniformly accuracy)