Moments of Vlasov-Poisson eq.

A journey from **Antonov** to **Morrison**

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Model problem

Model problem in dimension 1+1: $x \in I = [0,1]_{\mathrm{per}}$ and $v \in \mathbb{R}$,

$$\mathbf{V}.\mathbf{P}. \quad \begin{cases} \partial_t f + v\partial_x f - E\partial_v f = 0, & t > 0, \quad (x,v) \in I \times \mathbb{R}, \\ \partial_x E = \sqrt{2\pi} - \int_{\mathbb{R}} f dv, & t > 0, \quad x \in I. \end{cases}$$

Gauss equation is compatible with Ampère equation $\partial_t E = \int vfdv$ if the total momentum vanishes $\int_I \int_{\mathbb{R}} fvdv = 0$ at $t = 0$. So

$$\mathbf{V}.\mathbf{A}. \quad \begin{cases} \partial_t f + v\partial_x f - E\partial_v f = 0, & t > 0, \quad (x,v) \in I \times \mathbb{R}, \\ \partial_t E = \int_{\mathbb{R}} vfdv, & t > 0, \quad x \in I. \end{cases}$$

is the formulation used hereafter. Basic properties are

$$\begin{cases} \dfrac{d}{dt}\left[\int_I \int_{\mathbb{R}} f(t,x,v)v^2 dvdx + \int_I E(t,x,v)^2 dx\right] = 0, \\ 0 \le f(t,x,v) \le \|f_0\|_{L^\infty(I\times\mathbb{R})}, \qquad f(0,x,v) = f_0(x,v). \end{cases}$$

The initial data of **V.A.** always satisfies the Gauss law.

Linearization

We consider $f = G + g$, with $G(v) = \exp(-\frac{v^2}{2})$, so that $\int G(v)dv = \sqrt{2\pi}$ and the electric field is already firt order.
Linearized equations are

$$\begin{cases} \partial_t g + v\partial_x g & = -EvG, \quad t > 0, \quad (x,v) \in I \times \mathbb{R}, \\ \partial_t E & = \int_{\mathbb{R}} vg dv, \quad t > 0, \quad x \in I. \end{cases}$$

The **Antonov** energy identity writes

$$\frac{d}{dt}\left(\int_I \int_{\mathbb{R}} \frac{g^2}{G} dvdx + \int_I E^2 dx\right) = 0$$

In gravitational equations there is actually a minus: $-\int_I E^2$.

- Antonov: Remarks on the problem of stability in stellar dynamics, 1961
- Lemou-Mehats-Raphael: ≈ "Variational stability of gravitational systems", 2011
- Mouhot: Stabilité orbitale pour le système de Vlasov-Poisson gravitionnel, 2012

Symetrization

Set $M(v) = \sqrt{G(v)} = \exp(-\frac{v^2}{4})$.
The "good" unknown is: $(u, E) = \left(\frac{g}{M(v)}, E\right)$

$$\begin{cases} \partial_t u + v\partial_x u &= -vME, \quad t > 0, \quad (x, v) \in I \times \mathbb{R}, \\ \partial_t E &= \int_{\mathbb{R}} uvMdv, \quad t > 0, \quad x \in I, \end{cases}$$

with the energy identity $\frac{d}{dt}\left(\int_I \int_{\mathbb{R}} u^2 dvdx + \int_I E^2 dx\right) = 0$.

First goal is to analyze the structure of the underlying isometry group

$$(u_0, E_0) \longrightarrow \mathcal{L}(t)(u_0, E_0) = (u(t), E(t)) .$$

- Kato, Perturbation theory for linear operators, 1966.

Hermite polynomials

- Hilbert basis of the space $\int_{\mathbb{R}} f^2(v)G(v)dv < \infty$.
  - Rodrigue's representation $H_n(v) = (-1)^n G(v)^{-1} \frac{d^n}{dv^n} G(v)$.
  - The degree of H_n is n. The parity of H_n is the parity of n.
  - $\int H_n(v)H_m(v)G(v)dv = (2\pi)^{\frac{1}{2}} n!\ \delta_{nm}, \quad n, m \in \mathbb{N}$.
  - Recursion formula

$$H_{n+1}(v) = vH_n(v) - H_{n-1}(v).$$

- Hermite functions $\psi_n(v) = (2\pi)^{-\frac{1}{4}} n!^{-\frac{1}{2}} H_n(v)M(v)$ are more convenient:

$$\psi_0(v) = \frac{M(v)}{(2\pi)^{\frac{1}{4}}},\ \psi_1(v) = \frac{vM(v)}{(2\pi)^{\frac{1}{4}}},\ \psi_2(v) = \frac{(v^2-1)M(v)}{(8\pi)^{\frac{1}{4}}},\ \dots$$

Recursion: $v\psi_n(v) = \sqrt{n+1}\psi_{n+1}(v) + \sqrt{n}\psi_{n-1}(v)$.

- Hammett-Dorland-Perkins: Fluid models of phase mixing, Landau damping, and nonlinear gyrokinetics dynamics, 1992

Writing the moments

The moments of u are: $\alpha_n = \int_{\mathbb{R}} u\psi_n dv$

$$u(t,x,v) = \sum_{n\in\mathbb{N}} \alpha_n(t,x)\psi_n(v) \text{ with } \|u\|^2_{L^2(I\times\mathbb{R})} = \sum_{n\in\mathbb{N}} \|\alpha_n\|^2_{L^2(I)}.$$

Taking the moments of the evolution equation, the equations are now

$$\begin{cases} \partial_t\alpha_0 + \sqrt{n+1}\partial_x\alpha_{n+1} + \sqrt{n}\partial_x\alpha_{n-1} = -(2\pi)^{\frac{1}{4}}\delta_{n,1}E, \quad n = 0,1,2,\ldots \\ \partial_t E = (2\pi)^{\frac{1}{4}}\alpha_1. \end{cases}$$

Infinite linear system (infinite wave equation)

The infinite vector $U = \begin{pmatrix} E \\ \alpha_0 \\ \alpha_1 \\ \dots \end{pmatrix}$. satisfies: $\partial_t U + A\partial_x U = -iBU$

where the matrices are very sparse

$$A = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & 2^{\frac{1}{2}} & 0 & \dots \\ \hline 0 & 0 & 2^{\frac{1}{2}} & 0 & 3^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 3^{\frac{1}{2}} & 0 & 4^{\frac{1}{2}} \\ \dots & \dots & \dots & \dots & 4^{\frac{1}{2}} & \dots \end{array}\right)$$

and B is finite rank

$$B = \left(\begin{array}{ccc|ccc} 0 & 0 & -i(2\pi)^{\frac{1}{4}} & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ i(2\pi)^{\frac{1}{4}} & 0 & 0 & 0 & 0 & \dots \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \end{array}\right).$$

Same with Fourier

From $\widehat{U}_k = \widehat{U}(k) = \int_I e^{-ikx} U(x) dx$ for $k \in \mathbb{Z}$, one gets mode per mode

$$\partial_t \widehat{U}(k) = -i A_k \widehat{U}(k)$$

where A_k is an infinite matrix

$$A_k = \left(\begin{array}{ccc|ccc} 0 & 0 & -i(2\pi)^{\frac{1}{4}} & 0 & 0 & \dots \\ 0 & 0 & k & 0 & 0 & \dots \\ i(2\pi)^{\frac{1}{4}} & k & 0 & k2^{\frac{1}{2}} & 0 & \dots \\ \hline 0 & 0 & k2^{\frac{1}{2}} & 0 & k3^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & k3^{\frac{1}{2}} & 0 & k4^{\frac{1}{2}} \\ \dots & \dots & \dots & \dots & k4^{\frac{1}{2}} & \dots \end{array} \right)$$

Notice that A_k an unbounded auto-adjoint operator in $l_2 \equiv \left\{ W = (w_i)_{i \in \mathbb{N}}, \ \sum_{i \in \mathbb{N}} |w_i|^2 < \infty \right\}$.

Scattering theory says: compute the spectrum of kA, then add the finite rank perturbation $-iB$.

- Kato, Perturbation theory for linear operators, 1966.

Spectrum of A

The spectrum is made of two parts

$$S(A) = S_d(A) \cup S_c(A)$$

where

- $S_d(A) = \{0\}$ is pure spectrum with eigenvector is $\mathbf{e}_1 = (1, 0, 0, \dots)^t$.
- $S_c(A) = \mathbf{R}$ is continuous spectrum: generalized eigenvectors are U_λ for $\lambda \in \mathbb{R}$:

$$U_\lambda = \begin{pmatrix} 0 \\ \psi_0(\lambda) \\ \psi_1(\lambda) \\ \psi_2(\lambda) \\ \dots \end{pmatrix} \notin l_2.$$

The proof is by the recursion formula.

Spectrum of A_k, $k \neq 0$

The spectrum of A_k, $k \neq 0$, is

$$\mathrm{S}(A_k) = S_d(A_k) \cup S_c(A_k)$$

- $S_d(A_k) = \{0\}$: the eigenvector is $v_0^k = (k, -i\alpha, 0, 0, \dots)^t \in l^2$.
- $\mathrm{S}_c(A_k) = \mathbb{R}$: the generalized eigenvector associated to the eigenvalue λk is

$$V_\lambda^k = -iM(\lambda)ke_1 + \left(k^2 + P.V. \int_{\mathbb{R}} \frac{\mu}{\mu - \lambda} M(\mu)^2 d\mu\right) U_\lambda - M(\lambda) P.V. \int_{\mathbb{R}} \frac{\mu}{\mu - \lambda} U_\mu M(\mu) d\mu$$

where $P.V.$ indicates principal value.

- Stein: Singular integrals and differentiability of functions, Princeton Mathematical Series, 1970

Where does this strange formula come from?

Search for a formula

$$V_\lambda = a_\lambda \mathbf{e_1} + b_\lambda U_\lambda + P.V. \int_{\mathbb{R}} \frac{c_\lambda(\mu)}{\mu - \lambda} U_\mu d\mu$$

where a_λ, b_λ and $c_\lambda(\mu)$ are unknowns quantities.
The number of unknowns in this equation is "infinite" due to the unknown function c_λ.

The eigenvector equation $A_k V_\lambda = k\lambda V_\lambda$ is equivalent to

$$(A_k V_\lambda, \mathbf{e}_n) = k\lambda \left(V_\lambda, \mathbf{e}_n\right), \qquad \forall n \in \mathbb{N}$$

with

$$A_k = kA - iB$$

a sparse "tridiagonal" hermitian matrix with all coefficients known.

Trick: for all $n \geq P >> \text{rank}(B)$ it reduces to

$$(k\lambda U_\lambda, \mathbf{e}_n) + P.V. \int_{\mathbb{R}} \frac{c_\lambda(\mu)}{\mu - \lambda} (k\mu U_\mu, \mathbf{e}_n)\, d\mu$$

$$= (k\lambda U_\lambda, \mathbf{e}_n) + P.V. \int_{\mathbb{R}} \frac{c_\lambda(\mu)}{\mu - \lambda} (k\lambda U_\mu, \mathbf{e}_n)\, d\mu.$$

$$\Longleftrightarrow kP.V. \int_{\mathbb{R}} c_\lambda(\mu)\psi_n(\mu) d\mu = 0, \quad \forall n \geq P$$

$$\Longleftrightarrow c_\lambda \in \text{Span}(\psi_m)_{m<P}.$$

Therefore one has to deal only with a finite number of coefficients!! The solution is after that easy to compute.

The form of the final formula is (of course) very similar to the method of normal modes of Van Kampen
- Van Kampen: On the theory of stationary waves in plasmas, 1955
- K.M. Case, Plasma oscillations, Annals of Physics, 1959.
- Morrison and Pfirsch: Dielectric energy versus plasma energy, and Hamiltonian action-angle variables for the Vlasov equation, 1992.

Integro-differential operator

• The formula for the (V_λ) in terms of the (U_λ) is a kind of change of the "orthogonal" basis.

• Its enough to write directly its effect on $f(\lambda) = (W, U_\lambda)$ for an arbitrary $W \in l_2$. This is done with an integro-differential operator.

• The integro-differential operator writes

$$Lu = \left(-\partial_{xx} + q(v)\right) u(x,v) - vM(v)P.V. \int_{\mathbb{R}} \frac{1}{w-v} u(x,w)M(w)dw$$

where the function q is defined by

$$q(v) = P.V. \int_{\mathbb{R}} \frac{w}{w-v} M(w)^2 dw.$$

Up to the use of $u = \frac{g}{M}$ instead of $g = f - M^2$ it is the same $\mathcal{G}$ as in
- Morrison: Hamiltonian Description of Vlasov Dynamics: Action-Angle Variables for the Continuous Spectrum, 2000.

Main magic property

Take u that satisfies $\partial_t u + v\partial_x u + vME = 0$ with the Gauss law. Then $h = Lu$ is a solution of free transport

$$\partial_t h + v\partial_x h = 0.$$

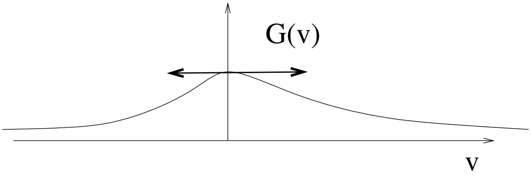
Direct proof is as follows

$$\begin{array}{lll} L\partial_t u & = \partial_t h & \\ L(v\partial_x u) & = v\partial_x Lu & -vM(v)\partial_x \int_{\mathbb{R}} u(x,w)M(w)dw \\ L(vM(v)E) & = & -vM(v)\partial_{xx} E \end{array}$$

The third column collapses thanks to the Gauss law.

Others

- The operator $u \mapsto q(v)u(x,v) - vM(v)P.V.\int_{\mathbb{R}} \frac{1}{w-v} u(x,w)M(w)dw$ is continous, Hilbert-Schmidt, triangular in the Hermite basis with almost zero diagonal.
- L is invertible: boundedness of L^{-1} in l_2 Fourier mode per Fourier mode,
- Exact formula for L^{-1} (Morrison, ...).
- The moment part of the proof is immediatly generalizable to "one bump" equilibrium



since $M^2(v) = -\frac{G(v)}{v} > 0$ is still positive.

Linear Landau damping with low regularity

• Consider a solution (u, E) of the linearized **V.P**. equations. Assume the initial condition is $u_0 \in L^2(I : H^n(\mathbb{R}))$ with $n \geq 1$. There exists a constant $C_n > 0$ such that

$$\|\phi(t)\|_{W^{1,\infty}(I)} + \|\partial_t \phi(t)\|_{L^\infty(I)} \leq \frac{C_n}{\max(1, t^n)} \|u_0\|_{L^2(I:H^n(\mathbb{R}))}.$$

where ϕ is the electric potential

$$-\partial_x \phi = E, \qquad \int_I \phi dx = 0.$$

• Key mechanism $\int_{\mathbf{R}} h_0(x - vt)\varphi(v)dv \approx \sum_{k\neq 0} \int_{\mathbb{R}} e^{-ikvt} f_k(v)dv$ with

$$\int_{\mathbb{R}} e^{-ikvt} f_k(v)dv = \frac{1}{(-ikt)^n} \int_{\mathbb{R}} e^{-ikvt} \frac{d^n}{dv^n} f_k(v)dv.$$

Classical proof *à la Landau* in
- Sonnendrucker: Modèles cinétiques pour la fusion, 2008.
- Mouhot and C. Villani, On Landau damping, 2011.

V. plus non linear **P.** equation

Non linear Vlasov-Poisson equation for ions

$$\begin{cases} \partial_t f + v\partial_x f + E\partial_v f = 0, & t > 0, \quad (x,v) \in I \times \mathbb{R}, \\ \lambda_D^2 \partial_x E = \int_{\mathbb{R}} f dv - n_e, & t > 0, \quad x \in I. \end{cases}$$

Here n_e is the density of electrons $E = -\partial_x \log n_e$, and $\lambda_D > 0$ is the Debye length: $n_e = \alpha^2 \exp\phi$ where ϕ is the elctric potential.
The interesting physical regime is when λ_D is small.
The formal limit $\lambda_D \to 0^+$ equation has been called the Vlasov-Dirac equation

$$\partial_t f + v\partial_x f - \partial_x \log\left(\int_{\mathbb{R}} f dv \log\right)\partial_v f = 0, \quad t > 0, \quad (x,v) \in I \times \mathbb{R}$$

known to be singular or close to be singular.

- Bardos, Nouri: A Vlasov equation with Dirac potential used in fusion plasmas, private.
- Jabin, Nouri: Analytic solutions to a strongly nonlinear Vlasov equation, 2011.

Linearization

Expansion $f = G + g$, $E = 0 + e$ and $n_e = \alpha^2(1 + \phi + O(\phi^2)$ yields

$$\begin{cases} \partial_t g + v\partial_x g + e\partial_v G = 0, & t > 0, \quad (x,v) \in I \times \mathbb{R}, \\ -\lambda_D^2 \partial_{xx}\phi + \alpha^2\phi = \int_{\mathbb{R}} g dv, & t > 0, \quad x \in I, \\ e = -\partial_x \phi, & t > 0, \quad x \in I. \end{cases}$$

The integro-differential operator becomes

$$L^D u(x,v) = \left(-\lambda_D^2 \partial_{xx} + \alpha^2 + q(v)\right) u(x,v)$$

$$-vM(v)P.V. \int_{\mathbb{R}} \frac{1}{w-v} u(x,w) M(w) dw.$$

- Much more in Morrison's papers

Linear damping of the field

Assume that $u_0 \in L^2(I : H^n(\mathbb{R}))$ with $\underline{n \geq 2}$.

Then the electric field tends to zero uniformly with respect to the Debye length with a rate at least t^{-n}.

Back to Antonov

The usual non linear equation

$$\mathbf{V.A.} \quad \begin{cases} \partial_t f + v\partial_x f - E\partial_v f = 0, & t > 0, \quad (x,v) \in I \times \mathbb{R}, \\ \partial_t E = \int_{\mathbb{R}} vfdv, & t > 0, \quad x \in I. \end{cases}$$

rewrites

$$\begin{cases} \partial_t u + v\partial_x u & +vME & = E\partial_v u - \frac{1}{2}vEu, \\ \partial_t E & -\int uvMdv & = 0. \end{cases}$$

The quadratic energy-**Antonov**' energy equation is

$$\frac{d}{dt}\left(\int_I \int_{\mathbb{R}} u^2 dxdv + \int_I E^2 dx\right) = -\int_I \left(E \int_{\mathbb{R}} vu^2 dv\right) dx.$$

The E is "gentle" (since bounded in L^∞) by the usual theory.
The v in $\int_{\mathbb{R}} vu^2 dv$ is the dangerous term.

- Rein: Collisionless Kinetic Equations from Astrophysics—The Vlasov-Poisson System, 2007

Trick

There is a "compatibility" between the left part of the system and its right part:

$$v\partial_x u + \frac{1}{2}vEu = ve^{\frac{1}{2}\phi}\partial_x\left(e^{-\frac{1}{2}\phi}u\right)$$

where ϕ is the electric potential $\partial_x\phi = -E$.

One more change of variable

Set

$$w(t,x,v) = e^{-\frac{1}{2}\phi(t,x)} u(t,x,v) \text{ and } F(t,x) = e^{-\frac{1}{2}\phi(t,x)} E(t,x)$$

which satisfy

$$\begin{cases} \partial_t w + v\partial_x w & +vMF & = E\partial_v w & -\frac{1}{2}\partial_t \phi w, \\ \partial_t F & -\int wvMdv & = & -\frac{1}{2}\partial_t \phi F. \end{cases}$$

The balance of the modified **Antonov**' energy writes

$$\frac{d}{dt}\mathcal{E}(t) = -\int \partial_t \phi \int w^2 dvdx - \int \partial_t \phi F^2 dx$$

where $\mathcal{E} = \frac{1}{2}\int\int |w|^2 dxdv + \frac{1}{2}\int |F|^2 dx$.

Quadratic energy

Only "gentle" terms in the RHS: $\|\partial_t \phi(t)\|_{L^\infty(I)} \leq C \min\left(1, \mathcal{E}(t)^{\frac{1}{2}}\right)$ since

$$\partial_x (\partial_t \phi) = -\partial_t E = -\underbrace{\int_{\mathbb{R}} vfdv}_{\|\cdot\|_{L}^{\frac{3}{2}}(I) \leq C} = -\underbrace{\int_{\mathbb{R}} uvMdv}_{\|\cdot\|_{L}^{2}(I) \leq \mathcal{E}^{\frac{1}{2}}}.$$

So: $\mathcal{E}' \leq C\mathcal{E} \times \min\left(1, \mathcal{E}^{\frac{1}{2}}\right)$.

Consider an initial data (u_0, E_0) (resp. (w_0, F_0)) with finite quadratic norm. Then $u(t)$ has finite quadratic norm for all time. Moreover $\mathcal{E}(t) \leq 2\mathcal{E}(0)$ for all $t \leq T_{**} = O\left(\mathcal{E}(0)^{-\frac{1}{2}}\right)$.

In order words: if the **Antonov**' energy is small initially, it remains approximatively constant during a quite long time.

Conclusion

These tools can be used ins some directions.

- Design test problems: low regularity initial data, preservation of quadratic norms, ...
- Truncate the system of moments and discretize it.
- The idea of moment has been discussed in some talks in the Workshop, for example for gyro-kinetic models and for the numerics.

- Helluy' talk: Numkin 2013.

Model

In view of numerical modeling, start from a a multiD **V.P.** model

$$\begin{cases} \partial_t f_i + v \cdot \nabla_x f_i + \dfrac{q_i}{m_i} \nabla_v \cdot [(\mathbf{E} + v \wedge \mathbf{B}_{\mathrm{ref}}) f_i] = 0, \\ \mathrm{div} \mathbf{E} = \dfrac{1}{\epsilon_0} \left(q_i \displaystyle\int f_i(v) dv - n_e \right), \\ \mathbf{E} = -\dfrac{k_B T_e}{q} \nabla_x \ln(n_e) \end{cases}$$

or with its gyrokinetic approximation.
Assume $\mathbf{B}_{\mathrm{ref}}$ is to be modeled to study the coupling. Since

$$\mathrm{curl} \mathbf{B} = \mu_0 \underbrace{q(n_I \mathbf{u}_I - n_e \mathbf{u}_e)}_{\text{electric current}} + \underbrace{\mu_0 \varepsilon_0}_{=1/c^2 \ll 1} \partial_t \mathbf{E}$$

$$\Rightarrow \mathbf{u}_e \approx \frac{1}{n_e} \left[n_I \mathbf{u}_I - \frac{\mathrm{curl} \mathbf{B}}{\mu_0 q} \right].$$

So one needs a companion equation which could be Ampère+Generalized Ohm's law.

Alternative in - Bonnito: Numkin 2013.

$$\partial_t \mathbf{B} + \mathrm{curl}\mathbf{E} = 0$$

$$n_e \mathbf{E} = -\frac{k_B T_e}{e}\nabla(n_e) + \underbrace{\left[\frac{\mathrm{curl}\mathbf{B}}{\mu_0 q} - n_I \mathbf{u}_I\right] \wedge \mathbf{B}}_{\text{comes from } -u_e \wedge \mathbf{B}} + n_e \underbrace{\eta \mathrm{curl}\mathbf{B}}_{\text{Joule effect}}$$

$$\begin{cases} -\lambda^2 \Delta \ln n_e = n_I - n_e, \\ \dfrac{\partial \mathbf{B}}{\partial t} - \mathrm{curl}\left(\dfrac{n_I \mathbf{u}_I}{n_e} \wedge \mathbf{B}\right) + \overbrace{\mathrm{curl}\left(\dfrac{\mathrm{curl}\mathbf{B}}{n_e} \wedge \mathbf{B}\right)}^{\text{Hall effect}} + \overbrace{\mathrm{curl}\left(\eta \mathrm{curl}\mathbf{B}\right)}^{\text{Joule effect}} = 0, \\ \dfrac{\partial f}{\partial t} + v \cdot \nabla f + \nabla_v \cdot \left[\left(-\nabla \ln n_e + \dfrac{\mathrm{curl}\mathbf{B} - n_I \mathbf{u}_I}{n_e} \wedge \mathbf{B} + v \wedge \mathbf{B}\right) f\right] = 0, \\ \mathrm{div}\mathbf{B} = 0. \end{cases}$$

Main point: Correct energy balance and existence of weak solution of the problem with physically sound boundary conditions.

Question: it is reasonnable from the physical viewpoint? At least λ_D small is necessary.

Can it be interesting for numerical models?

- Charles, D. Perthame, Sentis: Nonlinear stability of a Vlasov equation for magnetic plasmas, KRM 2013.

Thanks for your attention!