Conservative and positivity-preserving semi-Lagrangian kinetic schemes with spectrally accurate phase-space resolution

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Contents

Maxwell-Boltzmann system

Maxwell's equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \qquad \nabla \cdot \mathbf{B} = 0$$

Sources: charge and current density:

$$\rho\,(\mathbf{r},t) = \sum_\alpha q_\alpha\, n_\alpha\,(\mathbf{r},t)\,, \qquad \mathbf{J}\,(\mathbf{r},t) = \sum_\alpha q_\alpha\, n_\alpha\,(\mathbf{r},t)\,\mathbf{u}_\alpha\,(\mathbf{r},t)\,.$$

Number density and mean velocity of each species:

$$n_\alpha\,(\mathbf{r},t) = \int_{\mathbb{R}^3} f_\alpha\,(\mathbf{r},\mathbf{v},t)\,d\mathbf{v}, \qquad \mathbf{u}_\alpha\,(\mathbf{r},t) = \frac{1}{n_\alpha\,(\mathbf{r},t)} \int_{\mathbb{R}^3} \mathbf{v} f_\alpha\,(\mathbf{r},\mathbf{v},t)\,d\mathbf{v}.$$

Boltzmann's equation for each species:

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}\cdot\nabla f_\alpha + \frac{q_\alpha}{m_\alpha}\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)\cdot\nabla_{\mathbf{v}} f_\alpha \;=\; \sum_\beta Q_\alpha\left(f_\alpha, f_\beta\right)(\mathbf{r},\mathbf{v},t)$$

Boltzmann's equation

Eulerian formulation: $(t, \mathbf{x}, \mathbf{v})$ independent variables

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla f_\alpha + \frac{\mathbf{F}_\alpha}{m_\alpha} \cdot \nabla_{\mathbf{v}} f_\alpha = \left.\frac{\partial f_\alpha}{\partial t}\right|_{\text{coll}}$$

Lagrangian formulation: follow trajectory $(\mathbf{x}(t), \mathbf{v}(t))$ in phase space

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}\,(t), \qquad \frac{d\mathbf{v}}{dt} = \frac{1}{m_\alpha}\mathbf{F}_\alpha\,(t, \mathbf{x}(t), \mathbf{v}(t))$$

- Substituting into Boltzmann's equation: $\displaystyle \frac{Df_\alpha}{Dt} = \left.\frac{\partial f_\alpha}{\partial t}\right|_{\text{coll}}$
- Time rate of change of $f_\alpha\,(t, \mathbf{x}(t), \mathbf{v}(t))$ along phase-space trajectory only determined by collision operator
- Without collisions, f_α constant along phase-space trajectory: fluid motion in phase-space is **incompressible**

Semi-Lagrangian method:

- $f_\alpha(t, \mathbf{x}, \mathbf{v})$ lies on Eulerian mesh
- Evolution within time step uses Lagrangian formulation (*method of characteristics*)

Modeling challenges

WEAKLY COLLISIONAL PLASMA:

- Electrons can be far from equilibrium and involved in strongly non-linear processes (e.g. ionization near threshold)
- Multiple species: electrons, multiple ions, neutrals;
- Multiple time and spatial scales;
- Complex geometries, different boundary conditions (perfect/real conductors, dielectrics, absorbing), often time varying and coupled to domain (plasma feedbacks into circuit);
- Complex collisional processes: elastic, inelastic (excitation, ionization, recombination, attachment, dissociation etc.);
- External magnetic fields: electrons may be strongly magnetized, possibly ions too;
- Other important processes: radiation transport, gas-phase chemical reactions, plasma-surface interaction, aggregates (dusty plasmas).

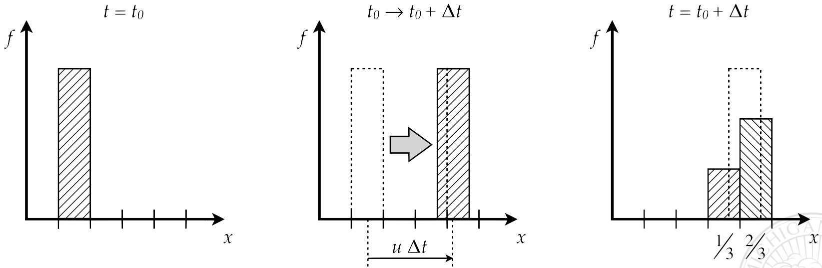
CHALLENGES FOR LOW-ORDER EULERIAN CODES:

- For electrons, need high resolution over large velocity mesh
- Impressive memory requirement in multiple dimensions
- Explicit time-stepping imposes non-physical time-step restriction (CFL limit)
- Method of lines (MOL): multistep and multi-stage methods require additional storage

Convected Scheme

The Convected Scheme [a] is a **forward semi-Lagrangian** method for Boltzmann's equation. Employs operator splitting:

1. **Collision operator** is local in configuration space, solves $\frac{\partial f_\alpha}{\partial t} = \left.\frac{\partial f_\alpha}{\partial t}\right|_{\text{coll}}$
2. **Ballistic operator** advects $f_\alpha(t, \mathbf{x}, \mathbf{v})$ along characteristic trajectories in phase space according to $\frac{Df_\alpha}{Dt} = 0$, integrated over a *moving cell* (MC).



$f_\alpha(t, \mathbf{x}, \mathbf{v})$ assumed uniform over MC, allowing for 'area remapping rule'

[a] W.N.G. Hitchon, D. Koch, and J. Adams. **An efficient scheme for convection-dominated transport**. *Journal of Computational Physics*, 83(1): 79-95, 1989.

Convected Scheme

PROs:

- Preserves positivity (good as $f_\alpha > 0$)
- No CFL restriction on Δt
- Very simple implementation
- Can enforce total energy conservation for stationary electric field

CONs:

- Difficult to handle boundary conditions
- **Numerical diffusion**: local remapping error $O\left(\Delta x^2\right)$

Reduced Numerical Diffusion

Numerical diffusion mitigated by reducing remapping frequency ⇒ "long-lived moving cells" [a]. Recently [b], we devised a high-order version of the Convected Scheme, for **neutral gas** kinetics:

Model equation: uniform velocity advection: $n_t + u_0 n_x = 0$

Basic idea: compensating remapping error by applying small corrections to final position of moving cells prior to remapping ⇒ **antidiffusive velocity field**

Tool: modified equation analysis, perturbation analysis

[a] A.J. Christlieb, W.N.G. Hitchon and E.R. Keiter. **A computational investigation of the effects of varying discharge geometry for an inductively coupled plasma**. *IEEE T. Plasma Sci.*, 28(6): 2214-2231, 2000.

[b] Y. Güçlü and W.N.G. Hitchon. **A high order cell-centered semi-Lagrangian scheme for multi-dimensional kinetic simulations of neutral gas flows**. *Journal of Computational Physics*, 231(8): 3289-3316, Apr 2012.

High-order semi-Lagrangian solution of the Vlasov-Poisson system

PROBLEM:

Difficult to construct high-order semi-Lagrangian ballistic operator when mean force is present (no straight trajectories)

SOLUTION:

- Further split ballistic operator into separate constant advection operators along x and v [a]
- Apply favorite high-order semi-Lagrangian solver to each operator
- Combine operators to high-order in time using Runge-Kutta-Nyström methods [b,c] (**symplectic** $\Rightarrow$ energy stable)

[a] C.Z. Cheng and G. Knorr. **The integration of the Vlasov equation in configuration space**. *J. Comput. Phys., 22: 330-351, 1976*.

[b] J.A. Rossmanith and D.C. Seal. **A positivity-preserving high-order semi-Lagrangian discontinuous Galerkin scheme for the Vlasov-Poisson equations**. *J. Comput. Phys., 227: 9527-9553, 2011*.

[c] N. Crouseilles, E. Faou and M. Mehrenberger. **High order Runge-Kutta-Nyström splitting methods for the Vlasov-Poisson equation**. *INRIA-00633934, 2011*.

Arbitrarily High-Order Convected Scheme (1)

1D CONSTANT ADVECTION EQUATION

$$\left(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x}\right) n(x,t) = 0$$

- Exact solution (method of characteristics): $n(x, t + \Delta t) \equiv n(x - u\Delta t, t)$
- Courant parameter: $\alpha := u\,\Delta t/\Delta x$

CONVECTED SCHEME UPDATE

- Discretize time (arbitrary Δt) and space (uniform Δx): $n_i^k \approx n(x_i, t_k)$
- Because of uniform Δx, solution can be shifted exactly by integer number of cells
- Without loss of generality, assume $0 \leq \alpha \leq 1$ (this is **not** a CFL limit)
- Under these assumptions, CS update is

$$n_i^{k+1} \; = \; U_{i-1}^k n_{i-1}^k + \left(1 - U_i^k\right) n_i^k$$

- As long as $0 \leq U_i^k \leq 1$, CS is **mass** and **positivity preserving**
- With no high-order corrections, $U(x,t) \equiv \alpha \quad \Rightarrow \quad$ 1st-order Upwind scheme
- With high-order corrections, $U(x,t) \; = \; \left[u + \tilde{u}(x,t)\right]\Delta t/\Delta x \; = \; \alpha + \tilde{\alpha}(x,t)$
- $\tilde{\alpha}(x,t)$ is **anti-diffusive Courant parameter**

Arbitrarily High-Order Convected Scheme (2)

LOCAL TRUNCATION ERROR (LTE)

- Exact solution, Taylor expand in space (smooth initial conditions):

$$n(x,t+\Delta t) \;=\; n(x,t) + \left(\sum_{p=1}^{N-1} (-\alpha)^p \frac{(\Delta x)^p}{p!} \frac{\partial^p}{\partial x^p} \right) n(x,t) + O\left(\Delta x^N\right),$$

- CS solution, Taylor expand in space about $(x,t) = (x_i, t^k)$:

$$n_{CS}(x,t+\Delta t) \;=\; n(x,t) + \left(\sum_{p=1}^{N-1} (-1)^p \frac{(\Delta x)^p}{p!} \frac{\partial^p}{\partial x^p} \right) U(x,t)\, n(x,t) + O\left(\Delta x^N\right)$$

- We want the local truncation error $\mathcal{E}(x,t,\Delta t) := n(x,t+\Delta t) - n_{CS}(x,t+\Delta t) \;=\; O(\Delta x^N)$, hence we find $\tilde{\alpha}(x,t)$ by imposing the **order condition**

$$\sum_{p=1}^{N-1} (-\alpha)^p \frac{(\Delta x)^p}{p!} \frac{\partial^p n}{\partial x^p} - \sum_{p=1}^{N-1} (-1)^p \frac{(\Delta x)^p}{p!} \frac{\partial^p (Un)}{\partial x^p} \;=\; O(\Delta x^N)$$

Arbitrarily High-Order Convected Scheme (3)

HIGH-ORDER CORRECTIONS [a]

- Make the polynomial ansatz $$Un(x,t) = \sum_{q=0}^{N-2} (-1)^q \, \beta_q(\alpha) \, (\Delta x)^q \, \frac{\partial^q n(x,t)}{\partial x^q},$$
  and solve for the unknown polynomials $\beta_q(\alpha)$;

- Substitute in order condition to find (after algebraic manipulations)
  $$\left[\tilde{\alpha} n\right]_i^k = \sum_{q=1}^{N-2} (-1)^q \, \frac{B_{q+1}(\alpha) - B_{q+1}(0)}{(q+1)!} \, (\Delta x)^q \, \frac{\partial^q n(x,t)}{\partial x^q}\bigg|_i^k,$$
  where $B_q(\cdot)$ are Bernoulli polynomials;

- Approximate products $(\Delta x)^q \, \frac{\partial^q n(x,t)}{\partial x^q}\Big|_i^k$ with error no larger than $O(\Delta x^{N-1})$, e.g.:
  1. linear polynomial interpolation,
  2. weighted essentially non-oscillatory (WENO) interpolation,
  3. fast Fourier transform (FFT).

[a] Y. GÜÇLÜ, A.J. CHRISTLIEB AND W.N.G. HITCHON. **Arbitrarily high order Convected Scheme solution of the Vlasov-Poisson system**. *In preparation*.

Arbitrarily High-Order Convected Scheme (4)

NUMERICAL IMPLEMENTATION [a]

- 6th-order finite difference scheme

$$\Delta x \; \frac{\partial n}{\partial x}\bigg|_i^k \approx \frac{n_{i-2}^k - 8\, n_{i-1}^k + 8\, n_{i+1}^k - n_{i+2}^k}{12} + O\Big(\Delta x^5\Big)\,,$$

$$(\Delta x)^2 \; \frac{\partial^2 n}{\partial x^2}\bigg|_i^k \approx \frac{-n_{i-2}^k + 16\, n_{i-1}^k - 30\, n_i^k + 16\, n_{i+1}^k - n_{i+2}^k}{12} + O\Big(\Delta x^6\Big)\,,$$

$$(\Delta x)^3 \; \frac{\partial^3 n}{\partial x^3}\bigg|_i^k \approx \frac{-n_{i-2}^k + 2\, n_{i-1}^k - 2\, n_{i+1}^k - n_{i+2}^k}{2} + O\Big(\Delta x^5\Big)\,,$$

$$(\Delta x)^4 \; \frac{\partial^4 n}{\partial x^4}\bigg|_i^k \approx n_{i-2}^k - 4\, n_{i-1}^k + 6\, n_i^k - 4\, n_{i+1}^k + n_{i+2}^k + O\Big(\Delta x^6\Big)\,,$$

- 22nd-order pseudo-spectral scheme

$$Un(x) = \mathcal{F}^{-1}\left[\sum_{q=0}^{\mathrm{N}-2} (-j)^q\, \beta_q(\alpha)\, (\xi \Delta x)^q \cdot \mathcal{F}[n](\xi)\right](x),$$

[a] Y. GÜÇLÜ, A.J. CHRISTLIEB AND W.N.G. HITCHON. **Arbitrarily high order Convected Scheme solution of the Vlasov-Poisson system**. *In preparation.*

Spectrally Accurate CS: Filtering

PROBLEM:

- Stable scheme for nonlinear Vlasov-Poisson must dissipate sub-cell features;
- Pseudo-spectral method has no dissipation mechanism, and preserves discrete L2-norm $\Rightarrow$ aliasing, Gibbs.

SOLUTION:

- Apply Fourier windowing in k-space [a] $\Rightarrow$ equivalent to Gaussian-regularized Sinc convolution in x (Gaussian has width $W\Delta x$);
- Adaptive filtering investigated in [a], but in our experience, need very accurate indicator to avoid excessive dissipation;
- We use constant $W = 4$ in all numerical examples $\Rightarrow$ low filter strength preserves spectral accuracy.

[a] Y. SUN, Y. ZHOU, S.-G. LI AND G. WEI. **A windowed Fourier pseudospectral method for hyperbolic conservation laws**. *J. of Comput. Phys., 214(2): 466-490, 2006*.

Spectrally Accurate CS: 1D Refinement Study

1D constant advection equation (normalized):

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial x} = 0, \quad x \in [-0.5, 0.5], \quad t \in [0, 1],$$

Smooth initial conditions:

$$n_0(0, x) \;=\; 0.5\, e^{-\left(\frac{x+0.2}{0.03}\right)^2} \;+\; e^{-\left(\frac{x}{0.06}\right)^2} \;+\; 0.5\, e^{-\left(\frac{x-0.2}{0.03}\right)^2}.$$

Refinement study:

N_x	**P4**	**Order**	**P6**	**Order**	**F22**	**Order**
32	1.41×10^{-01}	—	7.68×10^{-02}	—	2.47×10^{-02}	—
64	5.99×10^{-02}	1.24	2.55×10^{-02}	1.59	1.84×10^{-04}	7.07
128	2.28×10^{-02}	1.39	4.45×10^{-03}	2.52	7.55×10^{-11}	21.22
256	5.44×10^{-03}	2.07	2.14×10^{-04}	4.37	1.02×10^{-13}	9.53
512	7.94×10^{-04}	2.78	7.03×10^{-06}	4.93	1.20×10^{-13}	m.p.
1024	1.02×10^{-04}	2.96	2.21×10^{-07}	4.99	4.44×10^{-13}	m.p.
2048	1.28×10^{-05}	2.99	6.93×10^{-09}	5.00	4.20×10^{-13}	m.p.

Operator Splitting: ODEs

Separable ODE

$$\dot{u}(t) = A(u) + B(u)$$

Assume we can construct the exact solution to the subproblems

$$\dot{u}(t) = A(u) \qquad \dot{u}(t) = B(u)$$

in the form of the Lie operators

$$u_A(t+\Delta t) = e^{A\Delta t}u(t) \qquad u_B(t+\Delta t) = e^{B\Delta t}u(t)$$

If the operators do not commute, composition introduces splitting error. E.g.

Lie-Trotter $u(t+\Delta t) = e^{A\Delta t}e^{B\Delta t}u(t) + O(\Delta t^2)$

Leapfrog/Störmer-Verlet/Strang $u(t+\Delta t) = e^{A\Delta t/2}e^{B\Delta t}e^{A\Delta t/2}u(t) + O(\Delta t^3)$

NOTE 1: Composition of low-order schemes to get higher order.
NOTE 2: Most popular as symplectic integrators for Hamiltonian systems.

Operator Splitting: Linear Transport PDEs

Separable PDE: let $u = u(t, x)$, and $A(\cdot)$ and $B(\cdot)$ be integro-differential operators:

$$\frac{\partial u}{\partial t} = A(u) + B(u)$$

After discretization in x we have the approximate $e^{\tilde{A}\Delta t}$ and $e^{\tilde{B}\Delta t}$ where some error $O(\Delta x^p)$ was introduced.

Dimensional splitting for linear equations

$$\frac{\partial u}{\partial t} = a(x_2)\frac{\partial u}{\partial x_1} + b(x_1)\frac{\partial u}{\partial x_2} \qquad \text{with } u = u(t, x_1, x_2)$$

leads to 2 families of **constant advection equations**.

Hence, we can use our semi-Lagrangian solver for:

- 2D rotating advection
- linear Vlasov equation

Operator Splitting: Vlasov-Poisson (1)

Complication:
Time-varying electric field may reduce accuracy of splitting method!?

CHENG & KNORR'S ALGORITHM [a]

Based on Strang splitting, uses previous value of **E**, 2nd-order:

1. $\Delta t/2$ step on $(\partial_t + \mathbf{v}\cdot\nabla_{\mathbf{x}})\, f_e = 0$;
2. Compute $n_e = \int_{\mathbb{R}^3} f_e\, d\mathbf{v}$, solve $\nabla^2_{\mathbf{x}}\phi = \frac{q_e}{\varepsilon_0}(n_e - n_0)$, and evaluate $\mathbf{E} = -\nabla_{\mathbf{x}}\phi$;
3. Δt step on $(\partial_t + \frac{q_e}{m_e}\mathbf{E}\cdot\nabla_{\mathbf{v}})f_e = 0$;
4. $\Delta t/2$ step on $(\partial_t + \mathbf{v}\cdot\nabla_{\mathbf{x}})\, f_e = 0$.

NOTE: In [a] semi-Lagrangian advection (1D-1V, cubic splines in x, trigonometric interpolation in v)

[a] C. CHENG, AND G. KNORR. **The integration of the Vlasov equation in configuration space**. *J. Comput. Phys.*, 22(3): 330–351, 1976.

Operator Splitting: Vlasov-Poisson (2)

Problem: how to go higher order in time?

- High-order extrapolation of **E** and time averaging [a] (Lax-Wendroff / Cauchy-Kovalesky procedure)
- Use high-order symplectic integrators [b,c,d] based on Hamiltonian splitting $H = T + V$, with s stages:

$$\begin{cases} f_k^*(\mathbf{x},\mathbf{v}) \;=\; f_{k-1}\Big(\mathbf{x}-(a_k\Delta t)\mathbf{v},\mathbf{v}\Big) \\ f_k(\mathbf{x},\mathbf{v}) \;=\; f_k^*\Big(\mathbf{x},\mathbf{v}-(b_k\Delta t)\dfrac{q}{m}\mathbf{E}[f_k^*](\mathbf{x})\Big) \end{cases} \qquad (k=1,2,\ldots,s)\,,$$

[a] J.A. Rossmanith and D.C. Seal. **A positivity-preserving high-order semi-Lagrangian discontinuous Galerkin scheme for the Vlasov-Poisson equations**. *J. Comput. Phys., 227: 9527-9553, 2011*.

[b] T. Watanabe and H. Sugama. **Vlasov and drift-kinetic simulation methods based on the symplectic integrator**. *Trans. Th. Stat. Phys. 34: 287–309, 2005*.

[c] E. Pohn, M. Shoucri and G. Kamelander. **Eulerian Vlasov codes**. *Comput. Phys. Comm., 166(2): 81-93, 2005*.

[d] N. Crouseilles, E. Faou and M. Mehrenberger. **High order Runge-Kutta-Nyström splitting methods for the Vlasov-Poisson equation**. *INRIA-00633934, 2011*.

Operator Splitting: Vlasov-Poisson (3)

Methods compared:

Label	Description	Order	Stages	Refs.
LF2	Leap-frog / Strang / Störmer-Verlet	2	1	[a]
Y4	Triple-jump composition of LF2	4	3	[b],[c],[d]
O6-4	4th-order RKN, optimized	4	6	[e]
O11-6	6th-order RKN, optimized	6	11	[e]
O14-6	6th-order RKN, optimized	6	14	[e]

RKN = Runge-Kutta-Nyström

[a] G. Strang. **On the Construction and Comparison of Difference Schemes**. *SIAM J. Numer. Anal. 5(3): 506–517, 1968.*

[b] H. Yoshida. **Construction of higher order symplectic integrators**. *Phys. Lett. A, 150(5-7): 262-268, 1990.*

[c] E.Forest and R.D.Ruth. **Fourth-order symplectic integration**. *Physica D: Nonlinear Phenomena, 43(1): 105-117, 1990.*

[d] J. Candy and W. Rozmus. **A symplectic integration algorithm for separable Hamiltonian functions**. *J. Comput. Phys. 92(1): 230-256, 1991.*

[e] S. Blanes and P. Moan. **Practical symplectic partitioned Runge–Kutta and Runge–Kutta–Nyström methods**. *J. Comp. Appl. Math. 142(2): 313-330, 2002.*

2D Rotating Advection

Goal: characterize splitting error (closed orbits? phase error?)

- Model equation: $$\frac{\partial n}{\partial t} - (2\pi y)\,\frac{\partial n}{\partial x} + (2\pi x)\,\frac{\partial n}{\partial y} = 0$$
- Square domain: $(x, y) \in [-1, 1] \times [-1, 1]$
- Boundary conditions: periodic
- Initial conditions: superposition of two $(\cos)^{22}$ bells with elliptical cross-section

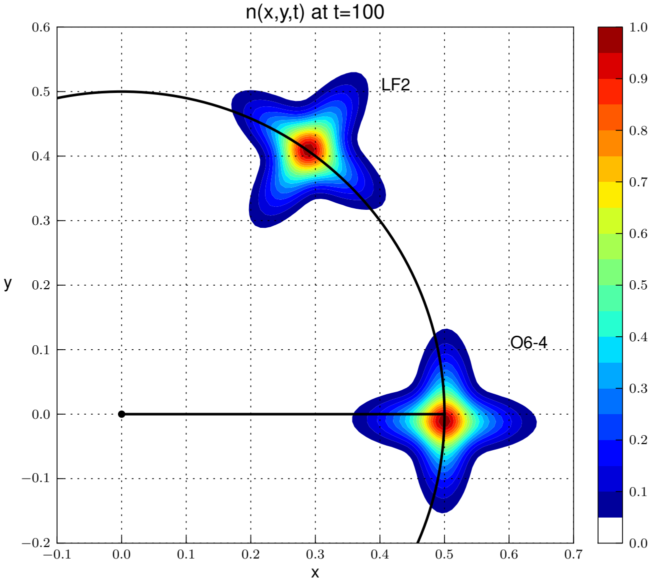
$$n(0, x, y) = 0.5\, B\big(r_1(x, y)\big) + 0.5\, B\big(r_2(x, y)\big),$$

$$B(r) = \begin{cases} \cos\left(\frac{\pi r}{2a}\right)^{22} & \text{if } r \leq a, \\ 0 & \text{otherwise,} \end{cases}$$

$$r_1(x, y) = \sqrt{(x - x_c)^2 + 8(y - y_c)^2},$$

$$r_2(x, y) = \sqrt{8(x - x_c)^2 + (y - y_c)^2},$$

2D Rotating Advection: phase-error



GRID

$N_x = 1024$
$N_v = 1024$

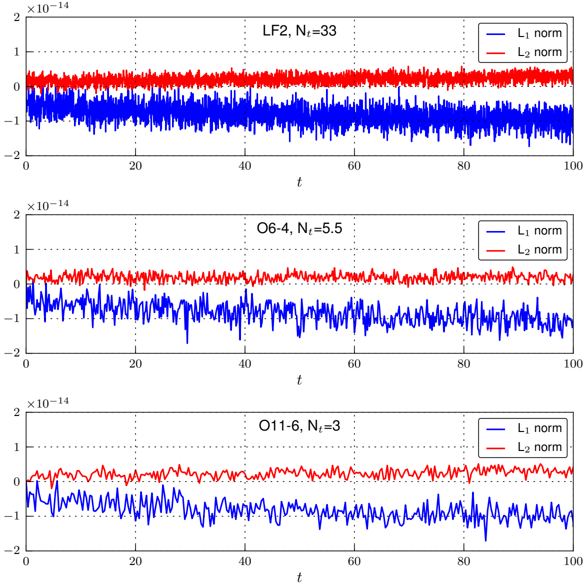
TIME STEPPING

$N_t(\text{LF2}) = 1/\Delta t = 33$
$N_t(\text{O6-4}) = 5.5$
$N_t(\text{O11-6}) = 3.0$

COURANT NUMBER

$C_p(\text{LF2}) \approx 195$
$C_p(\text{O6-4}) \approx 1170$
$C_p(\text{O11-6}) \approx 2145$

2D Rotating Advection: invariants / area preservation



GRID

$N_x = 1024$
$N_v = 1024$

TIME STEPPING

$N_t(\text{LF2}) = 1/\Delta t = 33$
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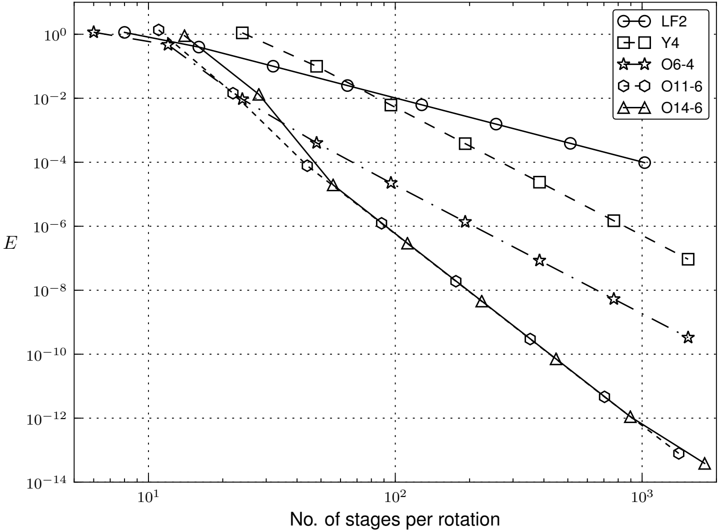
2D Rotating Advection: efficiency



GRID
$N_x = 1024$
$N_v = 1024$

Linear Vlasov Equation with Stationary Field

Goal: simulate trapping of electrons in stationary field
Challenges: fast filamentation, long-time confinement

- Model equation (normalized)

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - E(x)\frac{\partial f}{\partial v} = 0$$

- Electric field $E(x)$ given, not self-consistent
- Square domain: $(x, v) \in [-1, 1] \times [-1, 1]$
- Periodic boundary conditions in x and v (assume compact support)

- Special case of 2D advection equation in (x, v) coordinates
- Phase-space flow is incompressible: $\partial v/\partial x - \partial E/\partial v = 0$

Linear Vlasov: phase-space vorticity

- Prescribed stationary potential:

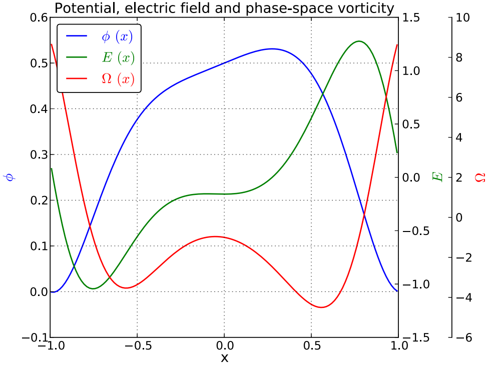
$$\phi(x) = \frac{1+\cos(\pi x^2)}{4} + \frac{\sin(\pi x)}{20}$$

- 2D flow field:

$$\mathbf{u}(\mathbf{x}) = \begin{bmatrix} u_1(x_1, x_2) \\ u_2(x_1, x_2) \end{bmatrix}$$

- 2D vorticity:

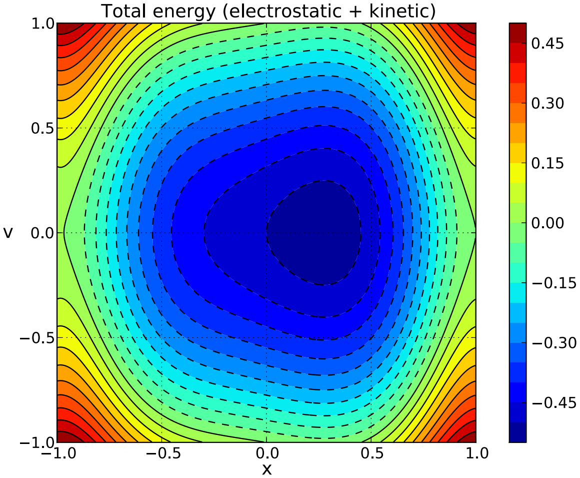
$$\nabla\times\mathbf{u} = \left[\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right]\hat{\mathbf{n}}_3 = \Omega(x_1, x_2)\,\hat{\mathbf{n}}_3$$



For 1D-1V Vlasov, $(x_1, x_2) = (x, v)$ and $(u_1, u_2) = (v, -E(x))$.
Hence the **phase-space vorticity** depends on x only:

$$\Omega(x, v) = -\frac{\partial E}{\partial x} - \frac{\partial v}{\partial v} = \frac{\partial^2\phi}{\partial x^2} - 1 = \Omega(x)$$

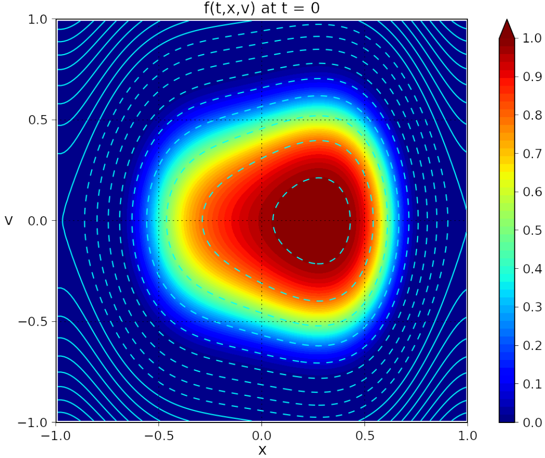
Linear Vlasov: Hamiltonian



- One-particle Hamiltonian:
  $$H(x, v) = \frac{v^2}{2} - \phi(x)$$
- Hamiltonian is constant of motion: $H(x(t), v(t)) \equiv c$
- $H(x, v) > 0$ (solid lines): open trajectories
- $H(x, v) \leq 0$ (dashed lines): closed trajectories (**electrostatic confinement**)

If Vlasov solver is **energy-stable**, confined particles will remain confined.

Linear Vlasov: steady-state preservation

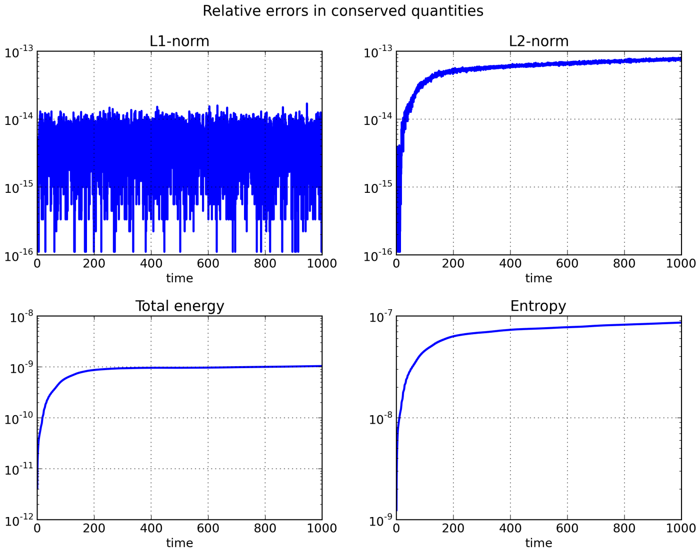


INITIAL CONDITIONS

$$f(0, x, v) = g\Big(H(x, v) - \min(H)\Big)$$

$$g(h) = \begin{cases} \cos(\pi h)^6 & \text{if } h < 0.5 \\ 0.0 & \text{otherwise} \end{cases}$$

Linear Vlasov: steady-state preservation



GRID

$N_x = 128$
$N_v = 128$

TIME STEPPING

$\Delta t = 0.25$
4000 steps

COURANT NO.

$C_x \approx 16$
$C_v \approx 20$

Linear Vlasov: filamentation (1)

(Loading Filamentation_coarse_animation.mp4)

GRID

$N_x = 512$
$N_v = 512$

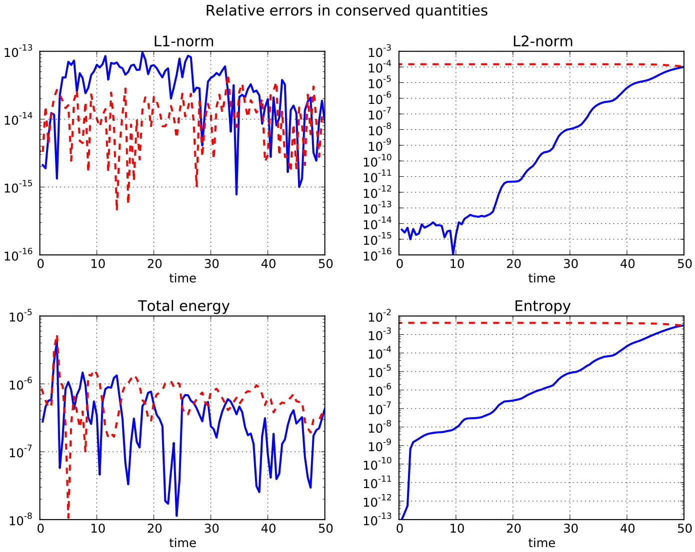
TIME STEPPING

$\Delta t = 0.5$
100+100 steps

COURANT NUMBER

$C_x \approx 130$
$C_v \approx 160$

Linear Vlasov: filamentation (2)



GRID

$N_x = 512$
$N_v = 512$

TIME STEPPING

$\Delta t = 0.5$
100+100 steps

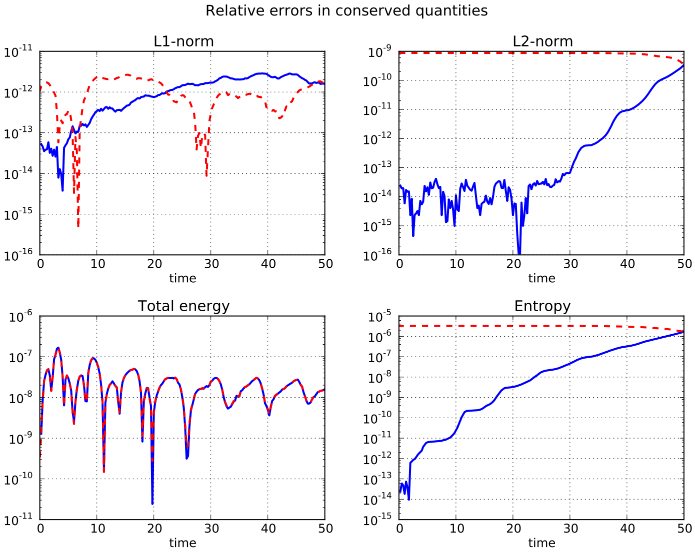
COURANT NO.

$C_x \approx 130$
$C_v \approx 160$

Solid blue line: **forward** evolution

Dashed red line: **backward** evolut.

Linear Vlasov: filamentation (3)



GRID

$N_x = 1024$
$N_v = 1024$

TIME STEPPING

$\Delta t = 0.25$
200+200 steps

COURANT NO.

$C_x \approx 130$
$C_v \approx 160$

Solid blue line:
forward evolution

Dashed red line:
backward evolut.

Vlasov-Poisson: bump-on-tail instability (1)

(Loading BumpOnTail_animation.mp4)

GRID

$N_x = 256$
$N_v = 512$

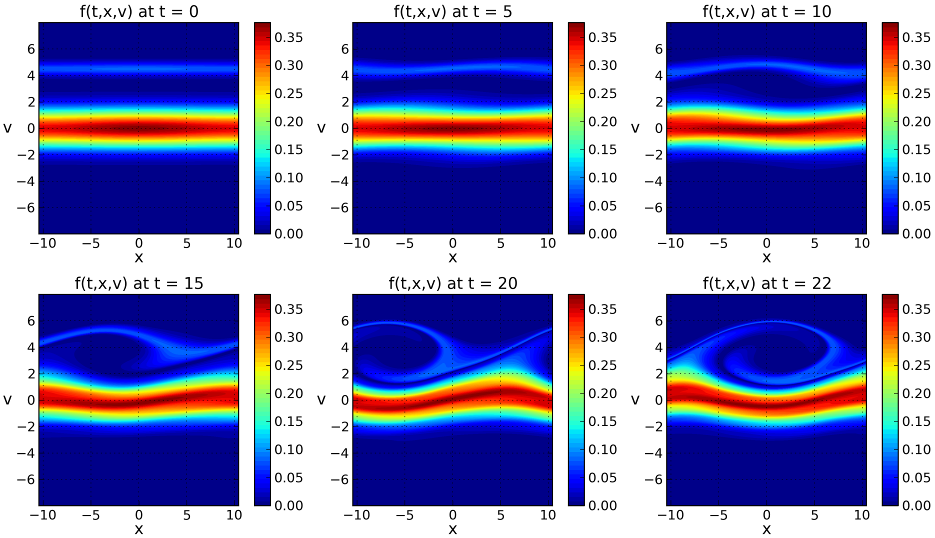
TIME STEPPING

$\Delta t = 0.5$
44 steps
O6-4

COURANT NUMBER

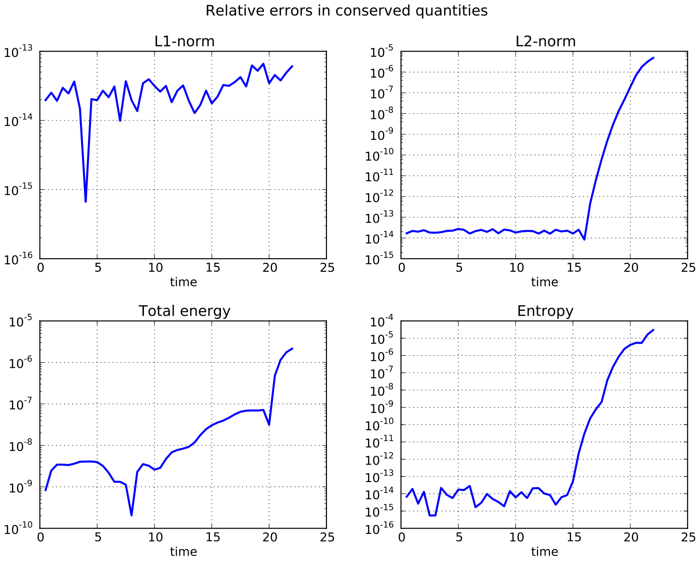
$C_x \approx 49$
$C_v \approx 9.4$

Vlasov-Poisson: bump-on-tail instability (2)



$N_x = 256 \quad N_v = 512 \quad \Delta t = 0.5 \quad$ 44 steps $\quad [C_x \approx 49 \quad C_v \approx 9.4] \quad$ O6-4

Vlasov-Poisson: bump-on-tail instability (3)



GRID

$N_x = 256$
$N_v = 512$

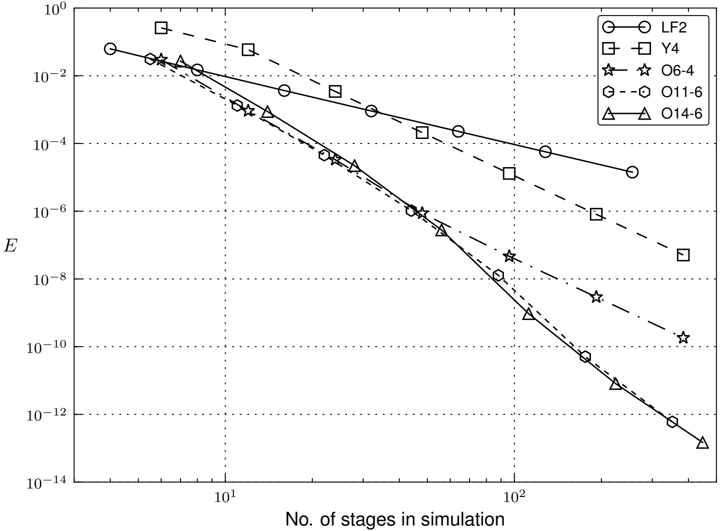
TIME STEPPING

$\Delta t = 0.5$
44 steps
O6-4

COURANT NO.

$C_x \approx 49$
$C_v \approx 9.4$

Vlasov-Poisson: bump-on-tail instability (4)



FINAL TIME

$T = 16$

GRID

$N_x = 1024$

$N_v = 1024$

REFERENCE

$N_x = 1024$

$N_v = 1024$

$\Delta t = 0.05$

O11-6

Vlasov-Poisson: linear Landau damping (1)

(Loading LinearLandau_animation.mp4)

GRID

$N_x = 16$
$N_v = 256$

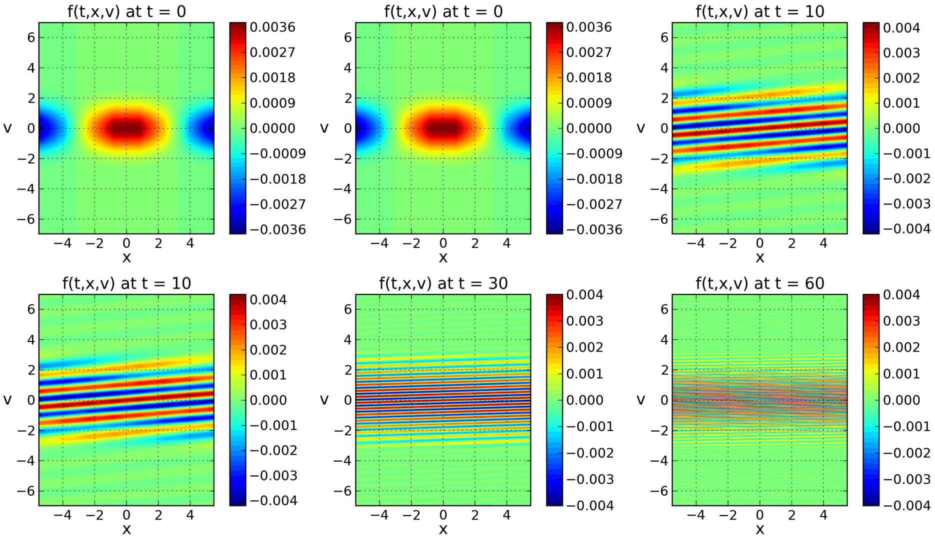
TIME STEPPING

$\Delta t = 0.5$
120 steps
O6-4

COURANT NUMBER

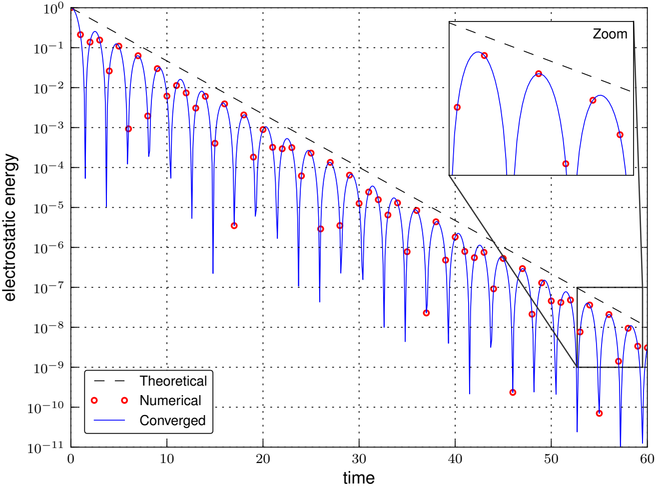
$C_x \approx 4.0$
$C_v \approx 0.2$

Vlasov-Poisson: linear Landau damping (2)



$N_x = 8$ $N_v = 256$ $\Delta t = 1.0$ 60 steps $[C_x \approx 4.0$ $C_v \approx 0.4]$ O6-4

Vlasov-Poisson: linear Landau damping (3)



GRID

$N_x = 8$

$N_v = 256$

TIME STEPPING

$\Delta t = 1.0$

60 steps

O6-4

COURANT NUMBER

$C_x \approx 4.0$

$C_v \approx 0.4$

REFERENCE

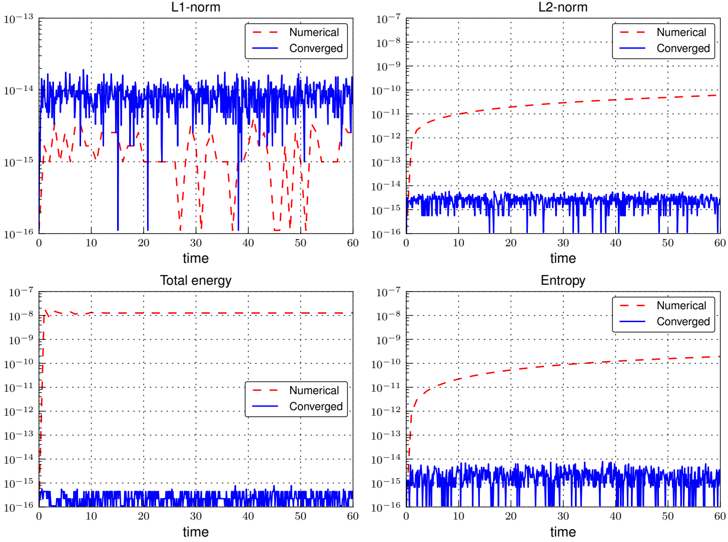
$N_x = 16$

$N_v = 512$

$\Delta t = 0.1$

O11-6

Vlasov-Poisson: linear Landau damping (4)



GRID

$N_x = 8$

$N_v = 256$

TIME STEPPING

$\Delta t = 1.0$

60 steps

O6-4

COURANT NO.

$C_x \approx 4.0$

$C_v \approx 0.4$

REFERENCE

$N_x = 16$

$N_v = 512$

$\Delta t = 0.1$

O11-6

Vlasov-Poisson: non-linear Landau damping (1)

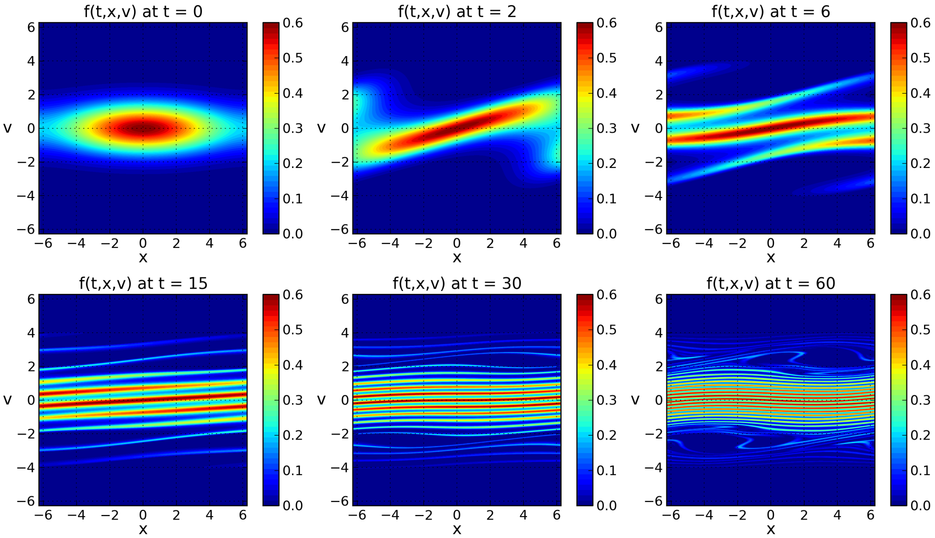
(Loading LinearLandau_animation.mp4)

GRID
$N_x = 256$
$N_v = 512$

TIME STEPPING
$\Delta t = 0.5$
120 steps

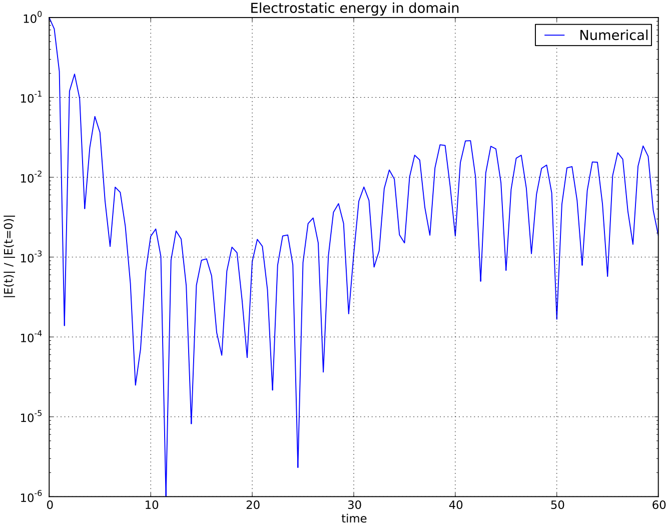
COURANT NUMBER
$C_x \approx 64$
$C_v \approx 20$

Vlasov-Poisson: non-linear Landau damping (2)



$N_x = 256 \quad N_v = 512 \quad \Delta t = 0.5 \quad$ 120 steps $\quad [C_x \approx 64 \quad C_v \approx 20]$

Vlasov-Poisson: non-linear Landau damping (3)



GRID

$N_x = 256$
$N_v = 512$

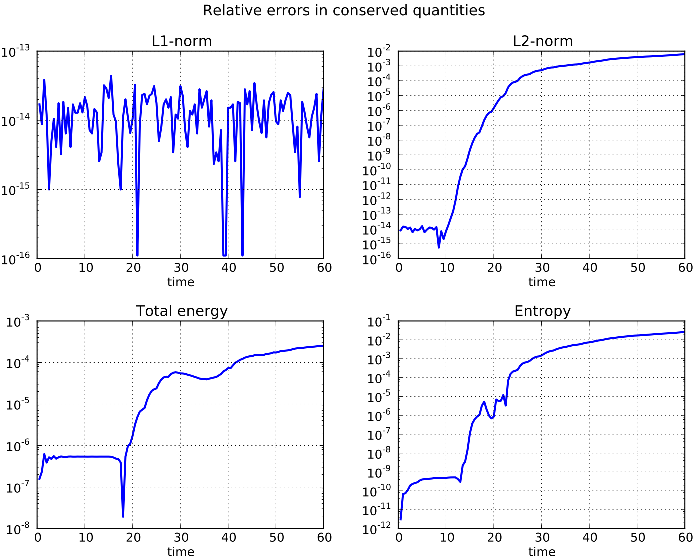
TIME STEPPING

$\Delta t = 0.5$
120 steps

COURANT NUMBER

$C_x \approx 64$
$C_v \approx 20$

Vlasov-Poisson: non-linear Landau damping (4)



GRID

$N_x = 256$

$N_v = 512$

TIME STEPPING

$\Delta t = 0.5$

120 steps

COURANT NO.

$C_x \approx 64$

$C_v \approx 20$

Vlasov-Poisson: two-stream instability (1)

(Loading TwoStream_animation.mp4)

GRID

$N_x = 256$
$N_v = 512$

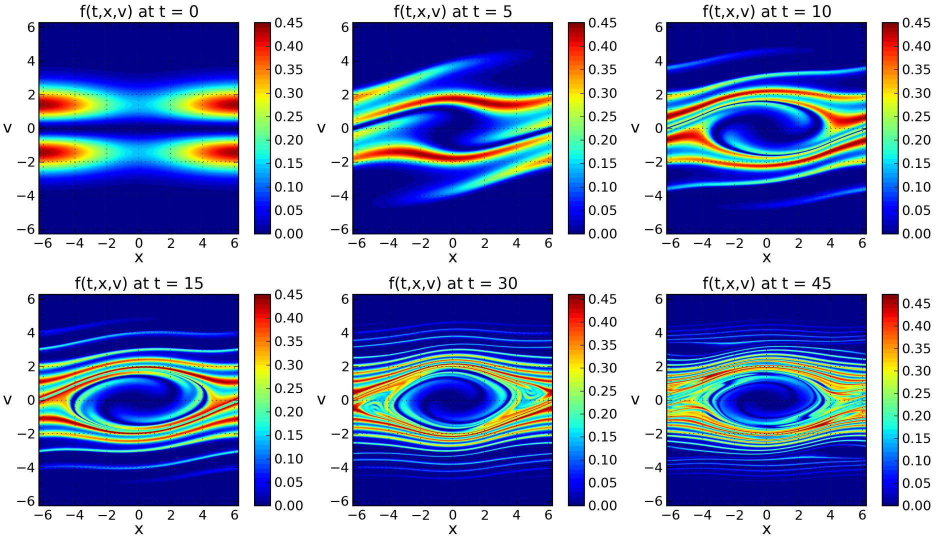
TIME STEPPING

$\Delta t = 0.5$
90 steps

COURANT NUMBER

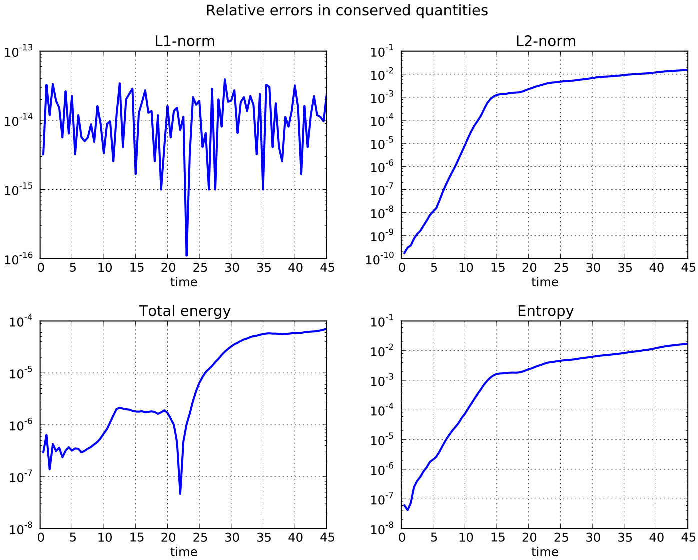
$C_x \approx 64$
$C_v \approx 20$

Vlasov-Poisson: two-stream instability (2)



$N_x = 256 \quad N_v = 512 \quad \Delta t = 0.5 \quad$ 90 steps $\quad [C_x \approx 64 \quad C_v \approx 20]$

Vlasov-Poisson: two-stream instability (3)



GRID

$N_x = 256$
$N_v = 512$

TIME STEPPING

$\Delta t = 0.5$
90 steps

COURANT NO.

$C_x \approx 64$
$C_v \approx 20$

Conclusions and Outlook

RECALL:

- Multi-dimensional mesh-based solution of Boltzmann's eq. for weakly collisional plasmas;
- In standard Eulerian codes, memory requirement may be too large (large mesh + RK storage) and time steps may be too small (CFL restriction);
- Splitting ballistic and collision operators allows semi-Lagrangian algorithms (no CFL limit);
- Splitting configuration-advection and velocity-advection permits one to use very accurate constant advection solvers (coarser mesh, no more RK storage).

SUMMARY:

- Convected Scheme (CS) is semi-Lagrangian algorithm, mass and positivity preserving;
- Constant advection CS extended to arbitrarily high order (22nd-order version with FFTs);
- High order (4th/6th) Runge-Kutta-Nyström operator splitting guarantees energy stability;
- Tested with standard benchmarks for 1D-1V Vlasov-Poisson system;
- Error in total energy conservation bounded until solution is spatially resolved.

WORK IN PROGRESS:

- Implement absorbing boundary conditions (wall recombination);
- Couple to simple collision operator (electron scattering on neutrals);
- Extend to higher dimensions (1D-2V, 2D-3V).

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