Reduced hyperbolic approximations of plasma models

Philippe Helluy, Laurent Navoret, Nhung Pham.

IRMA Strasbourg, Inria TONUS.

NumKin 2013, Garching.

Outlines

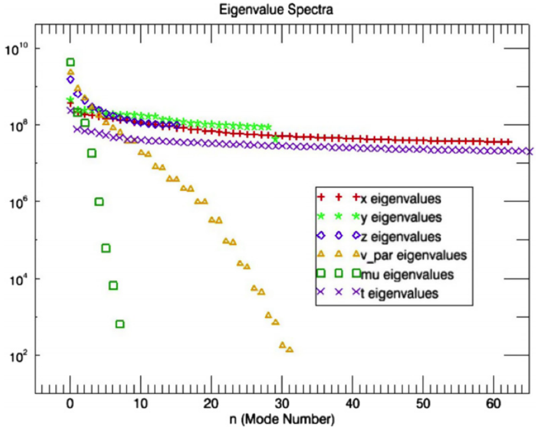
Context

- Transport kinetic models for tokamak core plasmas.
- Kinetic effects, plasma turbulence: fluid model not relevant.
- Possibly complex space geometry (aligned with the magnetic lines for instance), but rather simple velocity space geometry (typically a square).
- Different magnitudes of space and velocity turbulence.

Context

Hatch, del Castillo-Negrete, Terry, JCP 2012 [HCT12].

Data analysis on 5D gyrokinetic numerical simulations.



Mathematical model

We consider the one-dimensional Vlasov-Poisson model

$$\partial_t f + v\partial_x f + E\partial_v f = 0, \tag{1}$$

$$\partial_x E = -1 + \int_v f dv, \tag{2}$$

where $f(x,v,t)$ is the distribution function, $E(x,t)$ is the electric field.

We consider the space-periodic case

$$f(0,v,t) = f(L,v,t), \quad \frac{1}{L}\int_x\int_v f(x,v,0) = 1,$$

$$\int_{x=0}^{L} E dx = 0.$$

For the moment, we also suppose that

$$v \in]-\infty;\infty[.$$

Model reduction

We consider a finite number of independent velocity functions $\{\varphi_k(v), k = 1 \cdots P\}$ and expand f on this basis

$$f(x,v,t) \simeq \sum_{j=1}^{P} w^j(x,t)\varphi_j(v) = w^j(x,t)\varphi_j(v) \text{ (sum on repeated indices).} \quad (3)$$

The unknown scalar $f(x,v,t)$ is replaced by the unknown vector

$$w(x,t) = \left(w^1(x,t), w^2(x,t), \cdots, w^P(x,t)\right)^T.$$

Model reduction

We introduce the expansion (3) in the Vlasov equation (1), multiply by φ_i and integrate with respect to v. We obtain

$$M\partial_t w + A\partial_x w + B(E)w = 0, \tag{4}$$

where

$$M_{ij} = \int_v \varphi_i\varphi_j, \quad A_{ij} = \int_v v\varphi_i\varphi_j, \quad B(E)_{ij} = E\int_v \varphi_i\partial_v\varphi_j.$$

M is symmetric positive, A is symmetric. The system (4) is thus hyperbolic ($M^{-1}A$ has real eigenvalues). In addition, $B(E)$ is skew-symmetric, thus the entropy

$$\int_x\int_v w^T M w \simeq \int_x\int_v f^2$$

is constant with respect to time.

Model reduction

Some works using the same idea:

- Armstrong, 1967 [Arm67]
- Tang & *al.* 1992, 1993 [Tan93, TKR92]
- Schumer, Holloway, 1998 [SH98]
- le Bourdiec, de Vuyst, Jacquet 2006 [BVJ06]

and many more...

Numerical approximation

- We can use a standard hyperbolic PDE solver for approximating (4). The Poisson equation is solved by a standard elliptic solver.
- If the basis functions do not depend on x, the matrices M, A and B are constant.
- We can use an unstructured space approximation, such as Discontinuous Galerkin approximation.
- How to choose the basis function and the numerical method for achieving precision and efficiency?

Direct finite element interpolation

For practical reasons, we have first to bound the velocity space $v \in]-V, V[$.

Upwind condition at the boundaries

$$f(x, V, t) = 0 \text{ if } E(x,t) < 0 \text{ or } f(x, -V, t) = 0 \text{ if } E(x,t) > 0.$$

Vlasov equation, weak form [JP86]: find f such that for all continuous φ defined on $]-V, V[$,

$$\partial_t \int_v f\varphi + \partial_x \int_v vf\varphi + E \int_v \partial_v f\varphi$$
$$+\frac{E^+}{2} f(\cdot, -V, \cdot)\varphi(-V) - \frac{E^-}{2} f(\cdot, V, \cdot)\varphi(V) = 0, \tag{5}$$

with $E^+ = \max(E, 0)$ and $E^- = \min(E, 0)$.

Direct interpolation

First idea: use a d^{th} order nodal Lagrange interpolation finite element basis in v associated to nodes $(N_j)_{j=1\cdots P}$. As usual, the basis functions $(\varphi_i)_{i=1\cdots P}$ satisfy:

- φ_i is continuous, piecewise polynomial of degree d on $]-V,V[$,
- $\varphi_i(N_j)=\delta_{ij}$.

Low order: better sparsity of M, A and B. High order: better precision, but higher cost and condition numbers.

Direct interpolation

We obtain $M\partial_t w + A\partial_x w + B(E)w = 0$ with

$$M_{ij} = \int_v \varphi_i \varphi_j, \quad A_{ij} = \int_v v \varphi_i \varphi_j$$

and

$$B(E)_{ij} = \frac{E^+}{2}\varphi_j(-V)\varphi_i(-V) - \frac{E^-}{2}\varphi_j(V)\varphi_i(V) + \int_v \varphi_i \partial_v \varphi_j.$$

$B(E)$ is no more skew-symmetric. We have a small entropy dissipation arising from the boundaries

$$\frac{d}{dt}\int_x w^T M w = \frac{1}{2}E^- w_1^2 - \frac{1}{2}E^+ w_P^2 \leq 0.$$

Finite volume approximation

We consider the space step $\Delta x = L/N$, cell centers $x_i = (i+1/2)\Delta x$, $i = 0 \cdots N-1$. In cell i, the unknown vector $w(x,t)$ and electric field $E(x,t)$ are approximated by

$$w(x_i,t) \simeq w_i(t), \quad E(x_i,t) \simeq E_i(t).$$

Finite volume scheme

$$M\partial_t w_i = -\frac{F(w_i,w_{i+1}) - F(w_{i-1},w_i)}{\Delta x} - B(E_i)w_i.$$

$F(w_L,w_R)$ is the numerical flux. Time integration by a standard explicit scheme: Euler (first order) or Heun (second order).

Poisson: centered finite diffrences and FFT.

Numerical flux

Several possibilities

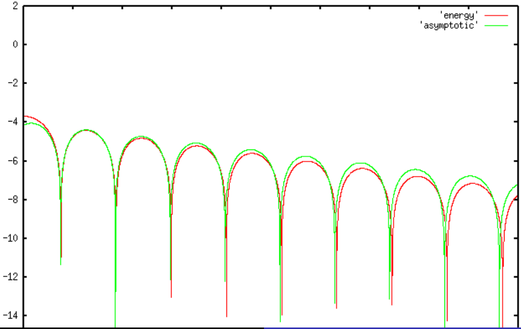
- Centered flux: $F(w_L, w_R) = \frac{1}{2}(Aw_L + Aw_R)$ (second order time integration required)
- Upwind flux: $F(w_L, w_R) = A^+ w_L + A^- w_R$.
- flux with small numerical viscosity $\kappa > 0$: $F(w_L, w_R) = \frac{1}{2}(Aw_L + Aw_R) - \frac{\kappa}{2}(w_R - w_L)$.

Properties of the scheme

- The first d moments of f are conserved (if $E = 0$ and $f(x, \pm V, t) = 0$).
- Stability under a standard CFL condition $V\Delta t \leq \Delta x$.
- M, A and B are sparse banded matrices. Mass lumping on M and A is possible.
- The centered scheme is second order.
- The numerical entropy remains constant with the centered scheme (if $f(x, \pm V, t) = 0$).

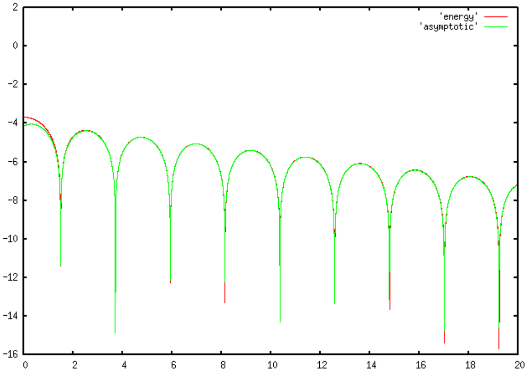
Landau damping

In this test case, we consider the following initial data
$f(x,v,0) = (1+\varepsilon cos(kx))\frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}, \ k = \frac{2\pi}{L}$.
We plot the log of the electric energy with respect to time
Upwind scheme, explicit Euler, $d=5, \ P=101, \ N=128, k=0.5$,
$\varepsilon = 5\times 10^{-2}$.



Second order

Centered flux and RK2 time integration.



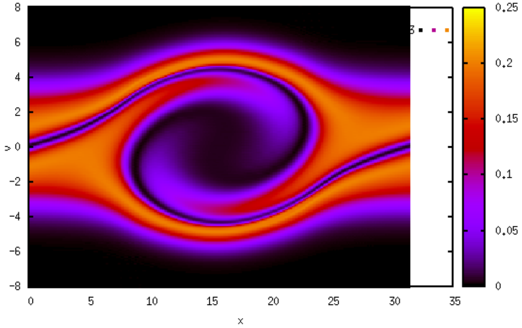
Two-stream instability

Initial data $f(x,v,0) = (1+\varepsilon cos(kx))\frac{1}{2\sqrt{2\pi}}\left(e^{-\frac{(v-v_0)^2}{2}} + e^{-\frac{(v+v_0)^2}{2}}\right)$,

$k = 0.2,\ v_0 = 3.$

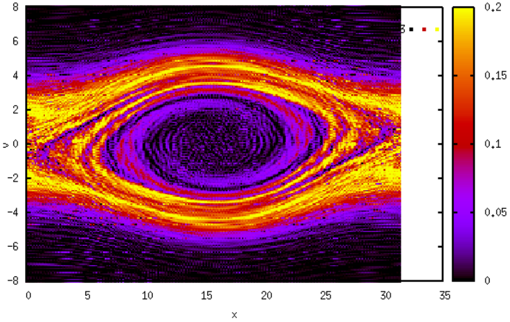
Centered flux, RK2, $d = 5$, $P = 101$, $N = 128$.

We plot the distribution function at $T = 25$



Two-stream instability

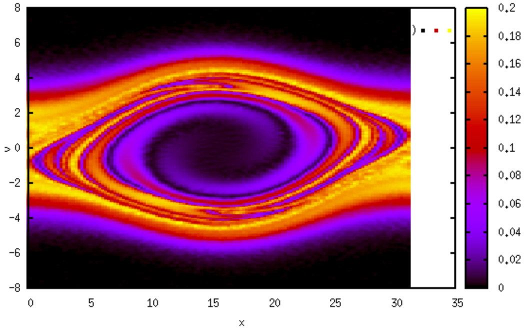
$T = 50$



Gibbs oscillations

Slightly upwind flux

$\kappa = 0.01$



Velocity Fourier transform

We can use another representation. Fourier transform with respect to v [Eli01]

$$\phi(x,\eta,t) = \int_v f(x,v,t)\exp(-Iv\eta), \quad I = \sqrt{-1}.$$

The Vlasov equation becomes

$$\partial_t \phi + I \partial_x \partial_\eta \phi + IE\eta\phi = 0.$$

Poisson equation

$$\partial_x E(x,t) = -1 + \phi(x,0,t).$$

Practical boundary condition: $\eta \in]-\eta_{\max}, \eta_{\max}[$, $\gamma > 0$,

$$\partial_x \phi(x, \pm\eta_{\max}, t) \pm I\gamma\phi(x, \pm\eta_{\max}, t) = 0.$$

Weak form

Weak form: find ϕ such that for all (continuous) φ

$$\int_\eta \varphi \partial_t \phi + \partial_x I \int_\eta \varphi \partial_\eta \phi + IE \int_\eta \eta \varphi \phi$$

$$-\frac{1}{2}\varphi(\eta_{\max}) I \partial_x \phi(\cdot, \eta_{\max}, \cdot) + \frac{1}{2}\varphi(-\eta_{\max}) I \partial_x \phi(\cdot, -\eta_{\max}, \cdot)$$

$$+\frac{1}{2}\varphi(\eta_{\max})\gamma\phi(\cdot, \eta_{\max}, \cdot) + \frac{1}{2}\varphi(-\eta_{\max})\gamma\phi(\cdot, -\eta_{\max}, \cdot) = 0. \quad (6)$$

Interpolation in Fourier space

We insert the expansion$\phi(x,\eta,t) = w^j(x,t)\varphi_j(\eta)$ in (6) and obtain

$$M\partial_t w + A\partial_x w + (B(E)+D)w = 0,$$

with

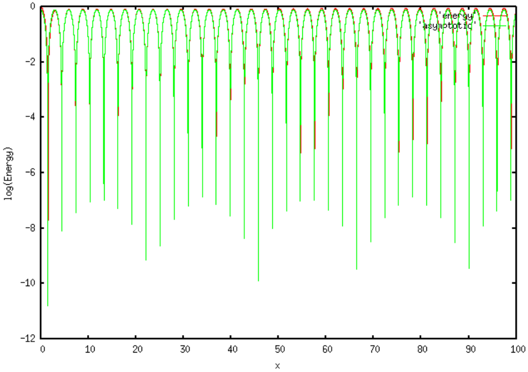
$$M_{ij} = \int_\eta \varphi_i\varphi_j,$$

$$B(E)_{ij} = IE\int_\eta \eta\varphi_i\varphi_j, \quad D_{ij} = \frac{1}{2}\gamma(\varphi_i\varphi_j(\eta_{\max}) + \varphi_i\varphi_j(-\eta_{\max}))$$

$$A_{ij} = I\int_\eta \varphi_i\varphi_j' - \frac{1}{2}I\varphi_i(\eta_{\max})\varphi_j(\eta_{\max}) + \frac{1}{2}I\varphi(-\eta_{\max})\varphi_j(-\eta_{\max}).$$

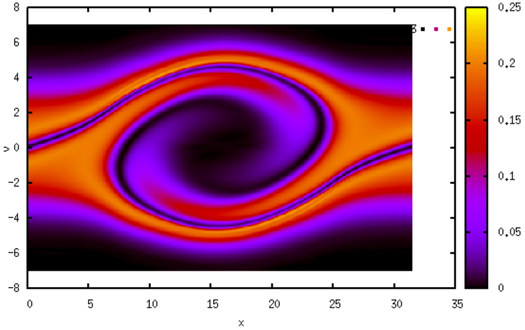
M is hermitian positive, $B(E)$ is skew-hermitian, D is diagonal positive and A is hermitian, thus the system is hyperbolic and entropy dissipative.

Landau damping



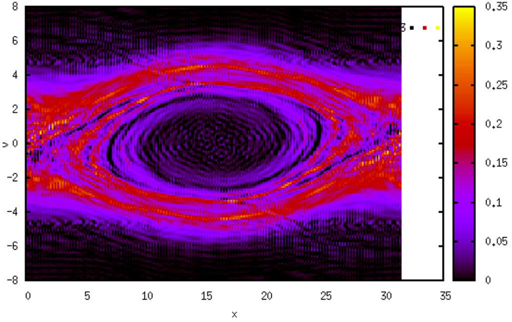
Two-stream instability

$T = 25$



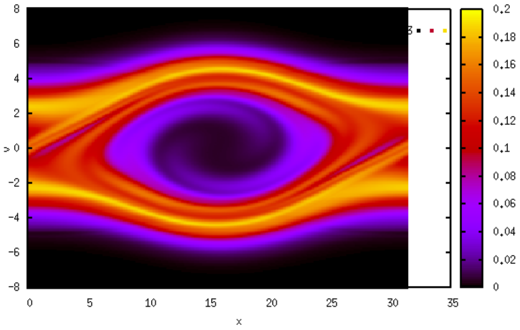
Two-stream instability

$T = 50$

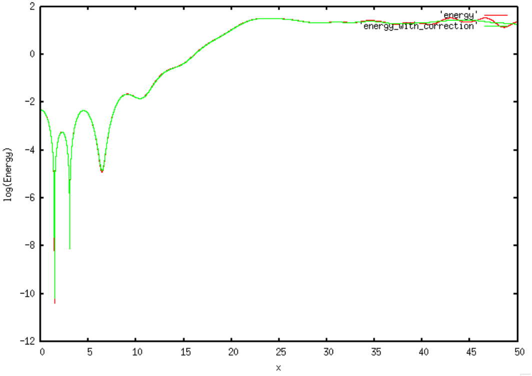


Dissipative flux

$\kappa = 0.05$



Energy


'energy'
'energy_with_correction'
log(Energy)
x
2
0
-2
-4
-6
-8
-10
-12
0
5
10
15
20
25
30
35
40
45
50

Non-linear model reduction

We can generalize the approach in order to conserve the physical entropy. We consider the Vlasov equation with a collision term

$$\partial_t f + v\partial_x f + E\partial_v f = Q(f).$$

We also consider a convex entropy $S(f)$. The entropy variable is then [2, 1, PDL09]

$$g = \partial_f S(f).$$

Considering the Legendre transform of S

$$S^*(g) = \max_f gf - S(f),$$

we also have

$$f = \partial_g S^*(g).$$

Entropy choice

Possible entropy choices

$$(a)\ S(f)=\frac{f^2}{2}.$$

With choice (a) we return to the linear case because

$$\partial_f S(f)=g=f.$$

$$(b)\ S(f)=f(\ln f-1),\quad (c)\ S(f)=\frac{f^2}{2}-\delta\ln f,\quad \delta>0.$$

Choice (b) corresponds to the physical entropy. Choice (b) ensures the positivity of the approximated f because $g=\ln f$ and thus $f=\exp(g)>0$.

Choice (c) also ensures positivity of f and can be interesting in some numerical reasons.

Interpolation in entropy variable

Instead of expanding f, we expand g on the interpolation basis

$$g(x,v,t) = \sum_{j=1}^{P} g_k(x,t)\varphi_k(v).$$

For modeling the collision, we can introduce the orthogonal projection Π on a subspace Λ_0 of the interpolation space $\Lambda = \text{span}\{\varphi_i, i = 1 \cdots P\}$. The collision model is, for $\lambda > 0$,

$$Q(f) = \lambda(\Pi g - g).$$

Physically, Λ_0 is a space of functions with low frequency oscillations in v. For instance

$$\Lambda_0 = \text{span}\{1, v, v^2\}.$$

Non-linear model reduction

We obtain

$$M(g)\partial_t g + A(g)\partial_x g + EB(g)g = Q,$$

with

$$M(g)_{ij} = \int_v \partial_{gg} S^*(g)\varphi_i\varphi_j, \quad A(g)_{ij} = \int_v v\partial_{gg} S^*(g)\varphi_i\varphi_j.$$

M and A are symmetric. M is positive (because S^* is strictly convex). Hyperbolic system.

- Moment conservation: $\forall \varphi_0 \in \Lambda_0$, $\int_v Q(f)\varphi_0 = 0$. We deduce that we have conservation laws for some "moments" of f.
- Entropy dissipation: $\Sigma = \int_v S(f)dv$ satisfies

$$\partial_t \Sigma + \partial_x G(\Sigma) \leq 0, \quad \text{with } G(\Sigma) = \int_v vS(f).$$

- $f > 0$.

Conclusion

- Approximation of the Vlasov equation well adapted to standard hyperbolic solvers.
- Possibility of model reduction.
- Numerical stabilization without breaking the conservation properties.
- The non-linear version can be made positive, high order and entropy conservative.
- Future works: higher dimensions, DG solver, boundary conditions...

P.T. Armstrong, Numerical studies of the nonlinear Vlasov equation, Phys. Fluids 10 (6) (1967) 1269–1280.

T. Barth. On discontinuous Galerkin approximations of Boltzmann moment systems with Levermore closure. Comput. Methods Appl. Mech. Engrg. 195 (2006), no. 25-28, 3311–3330.

S. Le Bourdiec, F. de Vuyst, L. Jacquet. Numerical solution of the Vlasov-Poisson system using gen- eralized Hermite functions. Comput. Phys. Comm. 175 (2006), no. 8, 528–544.

G Poëtte, B Després, D Lucor. Uncertainty quantification for systems of conservation laws. JCP, 2009.

Eliasson, Bengt Outflow boundary conditions for the Fourier transformed one-dimensional Vlasov-Poisson system. J. Sci. Comput. 16 (2001), no. 1, 1–28.

- D. R. Hatch, D. del-Castillo-Negrete, P. W. Terry. Analysis and compression of six-dimensional gyrokinetic datasets using higher order singular value decomposition. JCP 2012.
- C. Johnson, J. Pitkäranta. An analysis of the discontinuous Galerkin method for a scalar hyperbolic equation. Math. Comp. 46 (1986), no. 173, 1–26.
- P. Helluy, N. Pham. Space-only hyperbolic approximation of the Vlasov equation. http://hal.inria.fr/hal-00797974/
- B. Perthame. Boltzmann type schemes for gas dynamics and the entropy property. SIAM J. Numer. Anal. 27 (1990), no. 6, 1405–1421.
- J.W. Schumer, J.P. Holloway, Vlasov simulations using velocity-scaled Hermite representations, J. Comput. Phys. 144 (1998) 626–661.

- T. Tang, The Hermite spectral method for Gaussian-type functions, SIAM J. Sci. Comput. 14 (3) (1993) 594–606.
- T. Tang, S. McKee, M.W. Reeks, A spectral method for the numerical solutions of a kinetic equation describing the dispersion of small particles in a turbulent flow, J. Comput. Phys. 103 (1992) 222–230.