An exponential integrator for highly oscillatory Vlasov-Poisson systems

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OUTLINE

1. Equations of interest - Motivations
2. Stiff ODEs
3. Numerical schemes and the new algorithm
4. Numerical results
5. Conclusion

Vlasov equation – beam in a focusing channel

Paraxial approximation

For $\varepsilon \longrightarrow 0$ solve

$$\begin{cases} \dfrac{\partial f^\varepsilon}{\partial t} + \dfrac{v}{\varepsilon}\dfrac{\partial f^\varepsilon}{\partial r} + \left(E^\varepsilon - \dfrac{r}{\varepsilon}\right)\dfrac{\partial f^\varepsilon}{\partial v} = 0, \\ \dfrac{1}{r}\dfrac{\partial (r\,E^\varepsilon)}{\partial r} = \displaystyle\int f^\varepsilon(t,r,v)\,dv \\ f^\varepsilon(t=0,r,v) = f_0(r,v). \end{cases} \qquad (1)$$

where

- $f^\varepsilon = f^\varepsilon(t,r,v)$ particles distribution function
- Time $t \in [0,T]$, Position $r > 0$, Velocity $v \in \mathbb{R}$

- $r \mapsto r/\varepsilon$ focusing external electric field
- $E^\varepsilon(t,r)$ self-consistent electric field

Vlasov equation – strong magnetic field

Drift-kinetic regime

For $\varepsilon \longrightarrow 0$ solve

$$\begin{cases} \partial_t f^\varepsilon + \mathbf{v} \cdot \nabla_{\mathbf{x}} f^\varepsilon + \left(\mathbf{E}^\varepsilon + \dfrac{1}{\varepsilon} \mathbf{v}^\perp \right) \cdot \nabla_{\mathbf{v}} f^\varepsilon = 0, \\ \mathbf{E}^\varepsilon (\mathbf{x}, t) = -\nabla_{\mathbf{x}} \phi^\varepsilon, \quad -\Delta_{\mathbf{x}} \phi^\varepsilon = \displaystyle\int_{\mathbb{R}^2} f^\varepsilon d\mathbf{v} - n_i, \\ f^\varepsilon (\mathbf{x}, \mathbf{v}, t = 0) = f_0 (\mathbf{x}, \mathbf{v}), \end{cases} \tag{2}$$

where

- $f^\varepsilon = f^\varepsilon(t, \mathbf{x}, \mathbf{v})$ Particles distribution function
- Position $\mathbf{x} = (x_1, x_2)$, Velocity $\mathbf{v} = (v_1, v_2)$, and $\mathbf{v}^\perp = (-v_2, v_1)$

- **Strong** and constant magnetic field in the x_3 direction
- $\mathbf{E}^\varepsilon(\mathbf{x}, t)$ evolves in the plane $\perp$ to the magnetic field.

Particle-In-Cell method

Dirac sum approximation for f^ε :

$$f^\varepsilon{}_{N_p}(t, r, v) = \sum_{k=1}^{N_p} \omega_k \, \delta(r - R_k(t)) \, \delta(v - V_k(t))$$

where N_p is the number of **macroparticles** and $\big(R_k(t), V_k(t)\big)$ is the macroparticle k moving along a characteristic curve of Vlasov eq.

$$R'(t) = \frac{1}{\varepsilon} V(t), \qquad R(0) = r_0$$
$$V'(t) = -\frac{1}{\varepsilon} R(t) + E(t, R(t)), \qquad V(0) = v_0$$

The same thing for $\big(\mathbf{X}_k(t), \mathbf{V}_k(t)\big)$

$$\mathbf{X}'(t) = \mathbf{V}(t), \qquad \mathbf{X}(0) = \mathbf{x}_0$$
$$\mathbf{V}'(t) = \frac{1}{\varepsilon}\mathbf{V}^\perp(t) + \mathbf{E}^\varepsilon(t, \mathbf{X}(t)), \qquad \mathbf{V}(0) = \mathbf{v}_0$$

Highly oscillatory solutions

- When $E \equiv 0$, the solution is

$$\left(\begin{array}{c} R(t) \\ V(t) \end{array} \right) = \mathcal{R}\left(\frac{t}{\varepsilon}\right) \left(\begin{array}{c} r_0 \\ v_0 \end{array} \right) \text{ where } \mathcal{R}(\tau) = \left(\begin{array}{cc} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{array} \right).$$

- When $\mathbf{E} \equiv 0$, the solution is

$$\mathbf{X}(t) = \mathbf{x}_0 + \varepsilon \mathbf{v}_0^{\perp} - \varepsilon \mathcal{R}\left(\frac{t}{\varepsilon}\right) \mathbf{v}_0^{\perp}$$
$$\mathbf{V}(t) = \mathcal{R}\left(\frac{t}{\varepsilon}\right) \mathbf{v}_0$$

  $\mathbf{x}_0 + \varepsilon \mathbf{v}_0^{\perp}$ is the guiding center.

- When the electric field not zero => **stiff** solutions (*i.e.* evolving on two disparate time scales)

Reference : Frénod - Salvarani - Sonnendrücker, *M3AS*, 2009.

As $\varepsilon \to 0$, f^ε two-scale converges to F, i.e. $\boxed{f^\varepsilon(t,r,v) \sim F(t,\frac{t}{\varepsilon},r,v)}$

where $\dfrac{\partial F}{\partial \tau} + v\,\dfrac{\partial F}{\partial r} - r\,\dfrac{\partial F}{\partial v} = 0$, meaning that

$$\boxed{\mathrm{F}\Big(t,\tau,r,v\Big) = G\Big(t,\mathcal{R}^\tau(r,v)\Big),} \quad \text{where}$$

$\mathcal{R}^\tau$ is a rotation in $\mathbb{R}^2$ and $G = G(t,q,u)$ is the solution to

$$\begin{cases} \dfrac{\partial G}{\partial t}(t,q,u) + \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \mathcal{R}^\tau\Big(0, E^0\big(t,\tau,\mathcal{R}_r^{-\tau}(q,u)\big)\Big)\,d\tau \cdot \nabla_{q,u} G(t,q,u) = 0 \\ G(t=0,q,u) = \dfrac{1}{2\pi} f_0(q,u) \quad \text{and} \quad \dfrac{1}{r}\dfrac{\partial(r\,E^0)}{\partial r} = \displaystyle\int G\big(t,\mathcal{R}^\tau(r,v)\big)\,dv \end{cases}$$

Gain : larger Δt may be used in a numerical scheme for G.

Homogenization – The two-scale limit - First model

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Gain : larger Δt may be used in a numerical scheme for G.

The two-scale limit - Second model

Reference : Frénod - Sonnendrücker, *Asympt. Anal.* 1998.

As $\varepsilon \to 0$, f^ε two-scale converges to F, where $\dfrac{\partial F}{\partial \tau} + \mathbf{v}^\perp \cdot \nabla_{\mathbf{v}} F = 0$ *i.e.*

$$\boxed{\mathrm{F}(t, \tau, \mathbf{x}, \mathbf{v}) = G\Big(t, \mathbf{x}, \mathcal{R}^\tau(\mathbf{v})\Big),}$$

where $G = G(t, \mathbf{x}, \mathbf{u})$ is the solution to

$$\begin{cases} \dfrac{\partial G}{\partial t} = 0 \\[2ex] G(0, \mathbf{x}, \mathbf{u}) = \dfrac{1}{2\pi} f_0(\mathbf{x}, \mathbf{u}) \quad \text{and some limit Poisson equation} \end{cases}$$

Our approach

Aim

1. Perform simulation of the models (Vlasov-Poisson for f^ε) with large time steps with respect to the oscillation $(2\pi\varepsilon)$.
2. The scheme to be uniformly accurate when ε goes to zero.

General Problem : Solve **stiff** ODEs where stifness arises from the **linear** term

$$y'(t) = \frac{1}{\varepsilon} L y(t) + F(t, y(t)).$$

Difficulties : **We look for** a numerical scheme to be stable and accurate for **any** initial condition and during **both** phases (fast and slow).

Drawbacks :
– explicit methods need very small time step (for stability)
– fully implicit methods are costly (slow).

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Stiff ODEs

Stiff ODEs and slow manifolds

$$\text{When } \varepsilon \ll 1 \text{ solve} \quad y'(t) = -\frac{1}{\varepsilon} y(t) + F(t, y(t)).$$

1. **In** $\mathbb{R}$: $y'(t) = -\frac{1}{\varepsilon} y(t) - \sin t$. The solution is

$$y(t) = \left(y_0 - \frac{\varepsilon^2}{\varepsilon^2+1}\right) e^{-t/\varepsilon} - \frac{\varepsilon}{\varepsilon^2+1}(\sin t - \varepsilon \cos t).$$

Thus, ON the slow manifold

$$y(t) = \frac{-\varepsilon}{\varepsilon^2+1} \sin t + \frac{\varepsilon^2}{\varepsilon^2+1} \cos t.$$

2. **In** $\mathbb{C}$: $y'(t) = \frac{i}{\varepsilon} y(t) + e^{it}$. The solution is

$$y(t) = e^{it/\varepsilon}\left(y_0 + \frac{i}{1-1/\varepsilon}\right) - \frac{i}{1-1/\varepsilon} e^{it}.$$

"The 'slow manifold' is that particular solution which varies only on the slow time scale ; the general solution to the ODE contains fast oscillations also." - J.P. Boyd

Exponential integrators

Huge literature

- M. Hochbruck, A. Ostermann, *Exponential integrators*, Acta Numer., 2010.

Consider 3 methods avoiding the small time step :

- *ImEx* (or Linearly Implicit method),
- *Integrating Factor* (Lawson 1967)
- *Exponential time differencing* (Certaine 1960).

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Exponential time differencing - History

ImEx and *Integrating Factor* methods are frequently used for solving stiff PDE's. *ETD* is less common but has been re-invented many times over the years.

- the term comes from "computational electrodynamics" (Holland 1994, Taflove 1995) : *ETD1*
- Certaine 1960 : ideas + multistep *ETD* methods of any order
- Nørsett 1969 : arbitrary order A-stable exponential integrator
- ...
- Cox & Matthews 2002 : formulas for *ETD Runge-Kutta* methods of order up to 4. *ETD* is superior over *ImEx* and *Integrating Factor*!
- Kassam & Trefethen 2005 : *ETDRK4* is tested against 5 other 4th order schemes on several PDEs.
- ...

ImEx and *Integrating Factor* methods

ImEx : implicit formula to advance the linear part
explicit formula to advance the nonlinear part.

- works well on the slow manifold.
- fails to capture the stiff behaviour.
- for A-stability, cannot extend beyond 2nd order.

Integrating Factor : multiply the ODE by $e^{-tL/\varepsilon}$:

$$(e^{-tL/\varepsilon}y)' = e^{-tL/\varepsilon}F(t, y(t))$$

or $\quad u' = e^{-tL/\varepsilon}F(t, e^{tL/\varepsilon}u)$ and use an explicit scheme.

- inaccurate for F slowly varying
- the stiff part is solved **exactly**.
- can extend to any order.

Exact solution to

$$(e^{-tL/\varepsilon}y)' = e^{-tL/\varepsilon}F(t, y(t))$$

is

$$y_{n+1} = e^{(\Delta t/\varepsilon)\,L}y_n + e^{(\Delta t/\varepsilon)\,L}\int_{t_n}^{t_{n+1}} e^{(t_n-\tau)/\varepsilon\,L}F(\tau, y(\tau))\,d\tau$$

- smaller errors than IF on the slow manifold.
- the stiff part is solved **exactly**.
- can extend to any order.

Examples of *Exponential time differencing* schemes

- by Multi-step methods

ETD1 : approx F on $[t_n; t_{n+1}]$ by F_n

$$y_{n+1} = e^{(\Delta t/\varepsilon)\,L} y_n + F_n\,\varepsilon(e^{(\Delta t/\varepsilon)\,L} - 1)$$

ETD2 : approx F on $[t_n; t_{n+1}]$ by $\tau \mapsto F_n + (\tau - t_n)(F_n - F_{n-1})/\Delta t$

$$\ldots$$

- by Runge-Kutta methods

ETD2RK : approx F on $[t_n; t_{n+1}]$ by $\tau \mapsto F_n + (\tau - t_n)(\widetilde{F}_{n+1} - F_n)/\Delta t$, where

$$\widetilde{F}_{n+1} = F(t_{n+1}, e^{(\Delta t/\varepsilon)\,L} y_n + F_n\,\varepsilon(e^{(\Delta t/\varepsilon)\,L} - 1))$$

ETD3RK : approx F on $[t_n; t_{n+1}]$ by a quadratic interpolant ...

Particular cases

1) $L = -1$: Solve

$$y'(t) = -\frac{1}{\varepsilon} y(t) + F(t, y(t))$$

- ETD is used only for computing precisely the rapid decay (when the initial condition is OFF the slow manifold).
- Implicit Euler works very well (out of the fast phase) even with big time steps w.r.t. ε.

2) $L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$: Solve

$$\begin{cases} R'(t) = \dfrac{1}{\varepsilon} V(t) \\ V'(t) = -\dfrac{1}{\varepsilon} R(t) + E(t, R(t)). \end{cases}$$

ETD computes exactly the fast oscillations. Implicit Euler drifts inward.

Fast oscillations case

- $L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \Rightarrow \quad e^{tL} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} =: \mathcal{R}(t).$
- $E(t, r)$ is either given by Poisson or by $E(t, r) = -r^3$.

The *ETD* is

$$\begin{pmatrix} R_{n+1} \\ V_{n+1} \end{pmatrix} = \mathcal{R}\Big(\frac{\Delta t}{\varepsilon}\Big) \begin{pmatrix} R_n \\ V_n \end{pmatrix} + \mathcal{R}\Big(\frac{\Delta t}{\varepsilon}\Big) \int_{t_n}^{t_{n+1}} \mathcal{R}\Big(\frac{t_n - \tau}{\varepsilon}\Big) \begin{pmatrix} 0 \\ E(\tau, R(\tau)) \end{pmatrix} d\tau$$

Tests with the approximation *ETD2RK* : linear interpolation of the slow integrand through t_n and t_{n+1} by using *ETD1* for the prediction at t_{n+1}.

$\Rightarrow$ inaccurate results

since for $\Delta t \geq 2\varepsilon$ errors are significant (especially for particles off the slow manifold). The beam do not spiral at the good rate.

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$$\begin{pmatrix} R_{n+1} \\ V_{n+1} \end{pmatrix} = \mathcal{R}\Big(\frac{\Delta t}{\varepsilon}\Big) \begin{pmatrix} R_n \\ V_n \end{pmatrix} + \mathcal{R}\Big(\frac{\Delta t}{\varepsilon}\Big) \int_{t_n}^{t_{n+1}} \mathcal{R}\Big(\frac{t_n - \tau}{\varepsilon}\Big) \begin{pmatrix} 0 \\ E(\tau, R(\tau)) \end{pmatrix} d\tau$$

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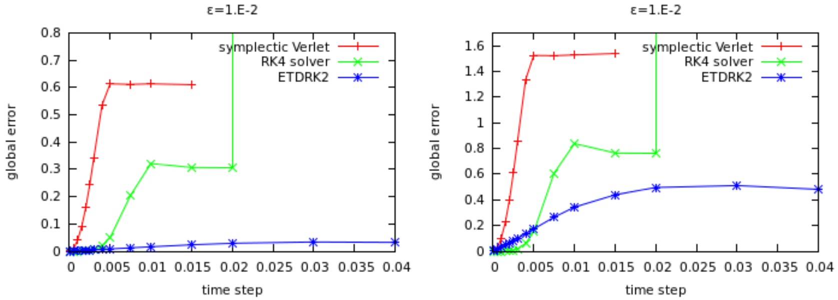
Global errors : the case $E(t, r) = -r^3$

$\varepsilon = 10^{-2}$, final time $= \pi$, size of the beam $\simeq 1$.
Starting with a particle **on** (left) and **off** (right) the slow manifold.



Similar results in the coupling with Poisson case.

The new ETD algorithm – big time steps

New algorithm (1/2)

If we want $\Delta t \gg \varepsilon$ then find the integer N and the real o s.t.

$$\Delta t = N \cdot (2\pi\varepsilon) + o$$

$2\pi\varepsilon$ is an approximation for the fast time for one grand tour.

Thus the integral term in the exact *ETD* writes

$$\int_{t_n}^{t_{n+1}} d\tau = \sum_{j=0}^{N-1} \int_{t_n+2\pi\varepsilon\, j}^{t_n+2\pi\varepsilon\,(j+1)} d\tau + \int_{t_{n+1}-o}^{t_{n+1}} d\tau$$

that we approximate by

$$N \int_{t_n}^{t_n+2\pi\varepsilon} d\tau + \int_{t_{n+1}-o}^{t_{n+1}} d\tau$$

New algorithm (2/2)

1. By the exact ETD we have

$$\mathcal{I}_1 = \int_{t_n}^{t_n + 2\pi\varepsilon} d\tau = \begin{pmatrix} R(t_n + 2\pi\varepsilon) - R(t_n) \\ V(t_n + 2\pi\varepsilon) - V(t_n) \end{pmatrix}.$$

2. we have to compute $\Big(R(t_n + N \cdot 2\pi\varepsilon), V(t_n + N \cdot 2\pi\varepsilon)\Big)$ since needed in the 3rd step. By the exact *ETD* we have

$$\begin{pmatrix} R(t_n + N \cdot 2\pi\varepsilon) \\ V(t_n + N \cdot 2\pi\varepsilon) \end{pmatrix} \simeq \begin{pmatrix} R_n \\ V_n \end{pmatrix} + N \cdot \mathcal{I}_1$$

3. likewise, $\mathcal{I}_2 = \int_{t_{n+1} - o}^{t_{n+1}} d\tau$ is

$$\mathcal{I}_2 = \mathcal{R}\Big(-\frac{o}{\varepsilon}\Big) \begin{pmatrix} \widetilde{R}(t_{n+1}) \\ \widetilde{V}(t_{n+1}) \end{pmatrix} - \begin{pmatrix} R(t_n + N \cdot 2\pi\varepsilon) \\ V(t_n + N \cdot 2\pi\varepsilon) \end{pmatrix}.$$

where $R(t_n + 2\pi\varepsilon)$ and $\widetilde{R}(t_{n+1})$ need to be calculated by RK4 with small step. Replacing these formulae in the exact *ETD* leads to

$$R_{n+1} = \widetilde{R}(t_{n+1}) \quad \text{and} \quad V_{n+1} = \widetilde{V}(t_{n+1})$$

The algorithm (ETD-PIC scheme)

The ODE's solution (R_n, V_n) at time t_n is given. Then

1. compute (R, V) at time $t_n + 2\pi\varepsilon$ by using a fine Runge-Kutta solver with initial condition (R_n, V_n).
2. compute (R, V) at time $t_n + N \cdot 2\pi\varepsilon$ by the following rule

$$\begin{pmatrix} R(t_n + N \cdot 2\pi\varepsilon) \\ V(t_n + N \cdot 2\pi\varepsilon) \end{pmatrix} = \begin{pmatrix} R_n \\ V_n \end{pmatrix} + N \begin{pmatrix} R(t_n + 2\pi\varepsilon) - R_n \\ V(t_n + 2\pi\varepsilon) - V_n \end{pmatrix}.$$

3. compute (R, V) at time t_{n+1} by using a fine Runge-Kutta solver with initial condition (R, V) obtained at the previous step.

Assumption :

$$\int_{t_n}^{t_n + N(2\pi\varepsilon)} \mathcal{R}\Big(\frac{t_n - \tau}{\varepsilon}\Big) \begin{pmatrix} 0 \\ E(\tau, R(\tau)) \end{pmatrix} d\tau \simeq N \cdot \int_{t_n}^{t_n + 2\pi\varepsilon} \ldots d\tau$$

Test case (2D phase space)

- initial condition for Vlasov :

$$f_0(r,v) = \frac{1}{\sqrt{2\pi}\, v_{th}} e^{-v^2/(2v_{th}^2)} \mathbf{1}_{[-R,R]}(r)$$

  where $v_{th} = 0.0727$ and $R = 0.75$.
- $N_p = 10000$ particles.

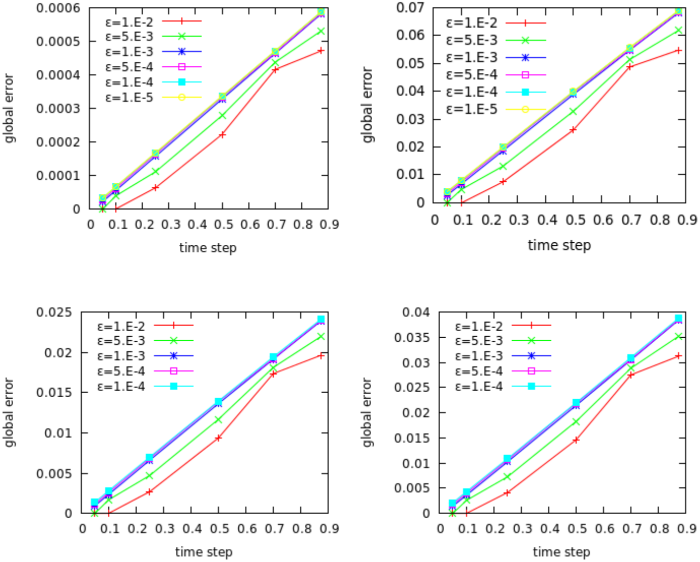
- $E(t,r) = -r^3$ or $-r$.
- coupling with Poisson equation

$$\frac{1}{r}\frac{\partial(r\,E)}{\partial r} = \int f(t,r,v)\,dv$$

  trapezoidal rule with 128 cells.

Global errors at final time 3.5

← ONsm i.c. $E = -r^3$ ↑ Poisson ↓ OFFsm i.c.→



A more accurate (**mean**) period

Use of an inaccurate period in the Algorithm can lead to instability

Example :

$$\begin{cases} R'(t) = \dfrac{1}{\varepsilon} V(t) \\ V'(t) = -\dfrac{1}{\varepsilon} R(t) - R(t). \end{cases}$$

- a phase space trajectory is an ellipse
- the rapid period is $T = 2\pi\varepsilon/\sqrt{1+\varepsilon}$ for **all** the initial particles => not spiraling beam.
- the slow manifold $\{(0,0)\}$
- ONsm i.c. $r_0 \sim 0.306$, $v_0 \sim 7 \cdot 10^{-6}$ and OFFsm i.c. $r_0 \sim 0.748$, $v_0 \sim 0.142$
- using $2\pi\varepsilon$ instead of T drifts particles outward in the phase space

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Global errors at final time 3.5 with the **mean** period

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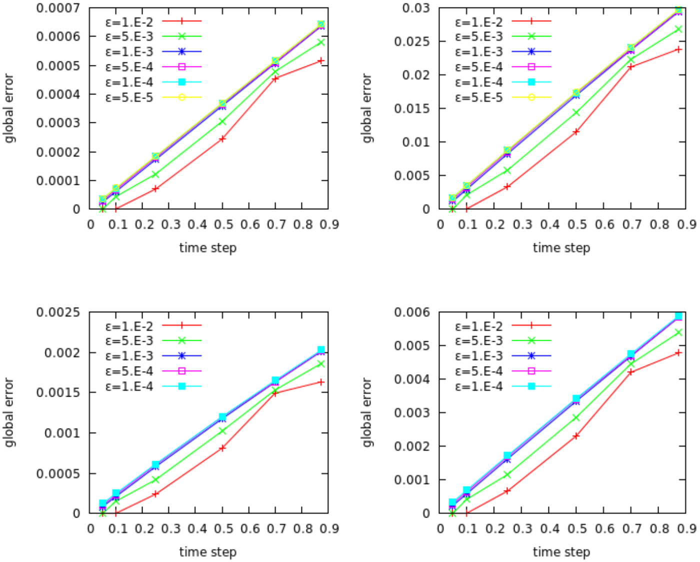
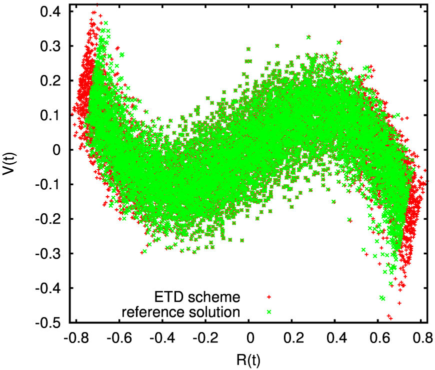
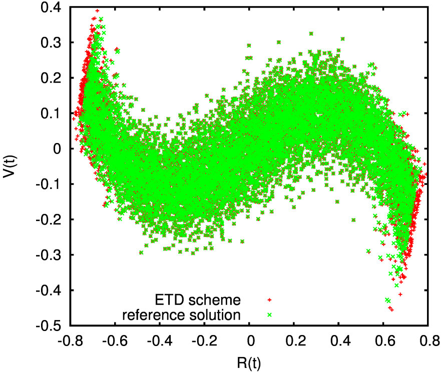


The case $E(t, r) = -r^3$

$\varepsilon = 10^{-4}$, $\boxed{\text{time step} = 8750\,\varepsilon}$, final time $= 3.5$, using for particles period $2\pi\varepsilon$ (at left) and the **mean** period (at right).





Vlasov-Poisson case

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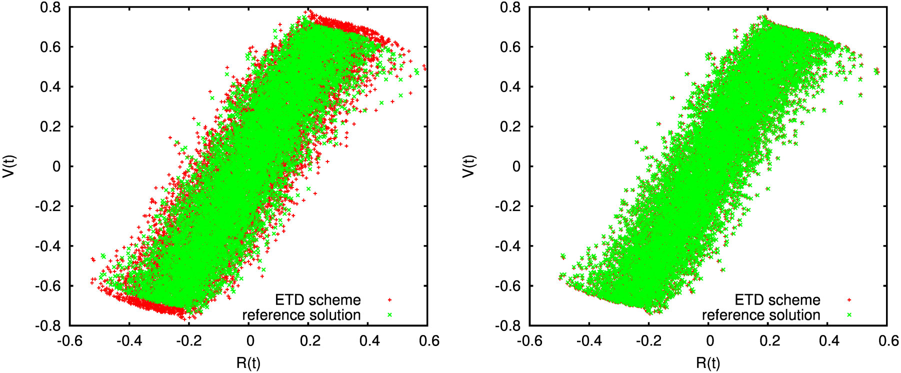

Second model – 4D phase space

$$\mathbf{X}'(t) = \mathbf{V}(t), \qquad \mathbf{X}(0) = \mathbf{x}_0$$
$$\mathbf{V}'(t) = \frac{1}{\varepsilon}\mathbf{V}^{\perp}(t) + \mathbf{E}^{\varepsilon}(t, \mathbf{X}(t)), \qquad \mathbf{V}(0) = \mathbf{v}_0$$

where $\mathbf{E}^{\varepsilon}(\mathbf{x}, t) = \begin{pmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 \end{pmatrix}$ or coupling with Poisson eq.

The **exponential integrator** in **velocity** :

- $\mathbf{V}(t) = e^{\frac{t-s}{\varepsilon}L}\,\mathbf{V}(s) + e^{\frac{t-s}{\varepsilon}L}\int_s^t e^{\frac{s-\tau}{\varepsilon}L}\,\mathbf{E}^{\varepsilon}(\mathbf{X}(\tau), \tau)\,d\tau.$
- $\mathbf{X}(t) = \mathbf{X}(s) + \int_s^t \mathbf{V}(\tau)\,d\tau.$

The algorithm (ETD-PIC scheme)

Find the integer N and the real o s.t.

$$\Delta t = N \cdot (2\pi\varepsilon) + o$$

$2\pi\varepsilon$ is an approximation for the fast time for one grand tour.
Assumption :

$$\int_{t_n}^{t_n+N(2\pi\varepsilon)} \mathcal{R}\Big(\frac{t_n-\tau}{\varepsilon}\Big)\mathbf{E}^{\varepsilon}\left(\mathbf{X}\left(\tau\right),\tau\right)d\tau \simeq N \cdot \int_{t_n}^{t_n+2\pi\varepsilon} \dots d\tau$$

The ODEs solution $(\mathbf{X}_n, \mathbf{V}_n)$ at time t_n is given. Then

1. compute $(\mathbf{X}, \mathbf{V})$ at time $t_n + 2\pi\varepsilon$ by using a fine Runge-Kutta solver with initial condition $(\mathbf{X}_n, \mathbf{V}_n)$.
2. compute $(\mathbf{X}, \mathbf{V})$ at time $t_n + N \cdot 2\pi\varepsilon$ by the following rule

$$\begin{pmatrix} \mathbf{X}(t_n + N \cdot 2\pi\varepsilon) \\ \mathbf{V}(t_n + N \cdot 2\pi\varepsilon) \end{pmatrix} = \begin{pmatrix} \mathbf{X}_n \\ \mathbf{V}_n \end{pmatrix} + N \begin{pmatrix} \mathbf{X}(t_n + 2\pi\varepsilon) - \mathbf{X}_n \\ \mathbf{V}(t_n + 2\pi\varepsilon) - \mathbf{V}_n \end{pmatrix}.$$

3. compute $(\mathbf{X}, \mathbf{V})$ at time t_{n+1} by using a fine Runge-Kutta solver with initial condition $(\mathbf{X}, \mathbf{V})$ obtained at the previous step.

The linear case

$$
\begin{aligned}
X_1^\varepsilon(t;\mathbf{x_0},\mathbf{v_0}) =& K_1^\varepsilon\left(\cos(a_\varepsilon t) - \frac{a_\varepsilon}{\varepsilon}\sin(a_\varepsilon t)\right) + K_2^\varepsilon\left(\sin(a_\varepsilon t) + \frac{a_\varepsilon}{\varepsilon}\cos(a_\varepsilon t)\right)\\
&+ K_3^\varepsilon\left(\cos(b_\varepsilon t) - \frac{b_\varepsilon}{\varepsilon}\sin(b_\varepsilon t)\right) + K_4^\varepsilon\left(\sin(b_\varepsilon t) + \frac{b_\varepsilon}{\varepsilon}\cos(b_\varepsilon t)\right),\\
X_2^\varepsilon(t;\mathbf{x_0},\mathbf{v_0}) =& -K_1^\varepsilon u_\varepsilon \cos(a_\varepsilon t) - K_2^\varepsilon u_\varepsilon \sin(a_\varepsilon t) - K_3^\varepsilon v_\varepsilon \cos(b_\varepsilon t) - K_4^\varepsilon v_\varepsilon \sin(b_\varepsilon t),\\
V_1^\varepsilon(t;\mathbf{x_0},\mathbf{v_0}) =& -K_1^\varepsilon a_\varepsilon\left(\frac{a_\varepsilon}{\varepsilon}\cos(a_\varepsilon t) + \sin(a_\varepsilon t)\right) + K_2^\varepsilon a_\varepsilon\left(\cos(a_\varepsilon t) - \frac{a_\varepsilon}{\varepsilon}\sin(a_\varepsilon t)\right)\\
&- K_3^\varepsilon b_\varepsilon\left(\frac{b_\varepsilon}{\varepsilon}\cos(b_\varepsilon t) + \sin(b_\varepsilon t)\right) + K_4^\varepsilon b_\varepsilon\left(\cos(b_\varepsilon t) - \frac{b_\varepsilon}{\varepsilon}\sin(b_\varepsilon t)\right),\\
V_2^\varepsilon(t;\mathbf{x_0},\mathbf{v_0}) =& K_1^\varepsilon a_\varepsilon u_\varepsilon \sin(a_\varepsilon t) - K_2^\varepsilon a_\varepsilon u_\varepsilon \cos(a_\varepsilon t) + K_3^\varepsilon b_\varepsilon v_\varepsilon \sin(b_\varepsilon t) - K_4^\varepsilon b_\varepsilon v_\varepsilon \cos(b_\varepsilon t),
\end{aligned}
$$

where $\underbrace{a_\varepsilon \sim \sqrt{3}\varepsilon}_{\text{slow motion}}$ and $\underbrace{b_\varepsilon \sim 1/\varepsilon}_{\text{fast motion}}$ and $(K_i^\varepsilon)_i$ depend on the i.c. $(\mathbf{x}_0, \mathbf{v}_0)$.

The slow manifold obtained when $K_3^\varepsilon(\mathbf{x}_0, \mathbf{v}_0) = 0$ and $K_4^\varepsilon(\mathbf{x}_0, \mathbf{v}_0) = 0 \Rightarrow$ a two dimensional space in $\mathbb{R}^4$.

Different initial particles

- Tests with 3 initial conditions

$$(\mathbf{x}_0^1, \mathbf{v}_0^1) = \left(1, 0, \varepsilon, -\frac{2\varepsilon u_\varepsilon}{2-\varepsilon^2} + \frac{\varepsilon^3 u_\varepsilon}{2-\varepsilon^2}\right) \quad \text{on the slow manifold}$$

$$(\mathbf{x}_0^2, \mathbf{v}_0^2) = \left(1, -\frac{u_\varepsilon}{w_\varepsilon}, \varepsilon\frac{w_\varepsilon}{u_\varepsilon}, -\frac{2\varepsilon u_\varepsilon}{2-\varepsilon^2} + \varepsilon\frac{w_\varepsilon}{u_\varepsilon} + \frac{\varepsilon^3 u_\varepsilon}{2-\varepsilon^2}\right) \quad \text{close to the slow manifold}$$

$$(\mathbf{x}_0^3, \mathbf{v}_0^3) = (1, 1, 1, 1) \quad \text{far from the slow manifold}$$

- Tests with a beam

$$f_0(\mathbf{x}, \mathbf{v}) = \frac{1}{8\pi^2 v_{th}^2}\left(1 + \eta\cos(\mathbf{k}\cdot\mathbf{x})\right)\chi(\mathbf{x})\exp\left(-\frac{v_1^2+v_2^2}{2v_{th}^2}\right),$$

with $k_1 = 0.5$, $k_2 = 0$, $v_{th} = 0.1$, $\eta = 0.1$, and

$$\chi(\mathbf{x}) = \chi_{[0,4\pi]}(x_1)\,\chi_{[0,1]}(x_2).$$

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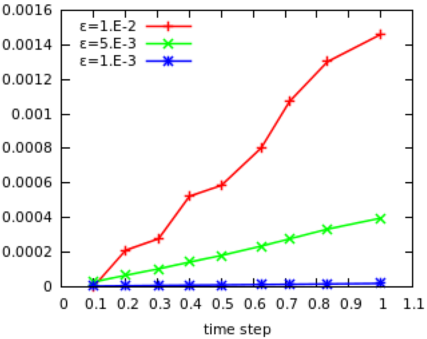
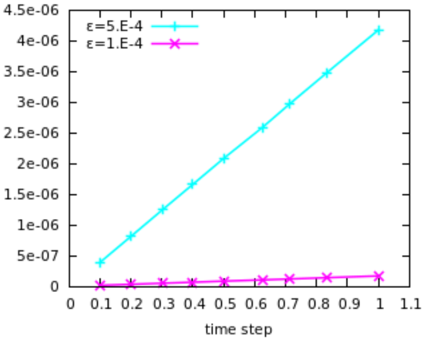
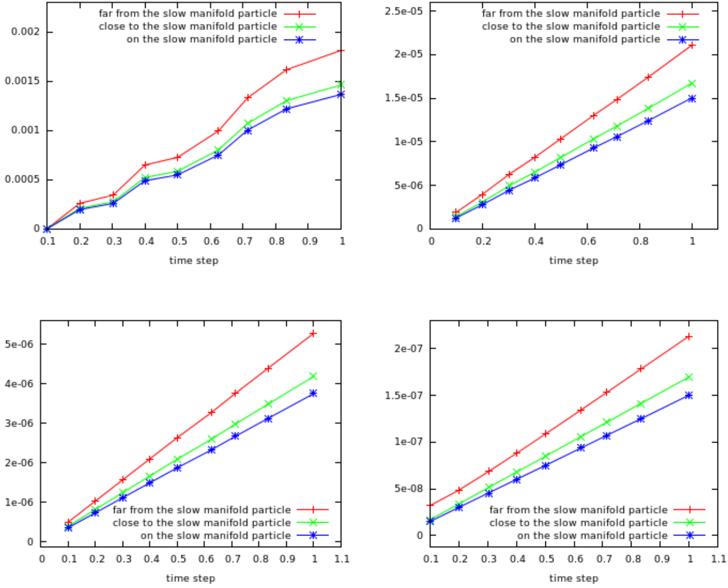
Global errors at final time 10

An initial condition **close** to the slow manifold





Global errors at final time 10 : $\varepsilon = 10^{-2},\ 10^{-3},\ 5 \cdot 10^{-4},\ 10^{-4}$

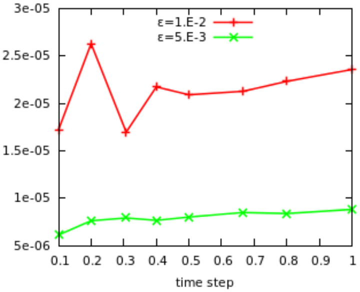


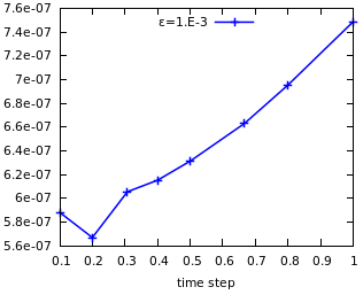
The Vlasov-Poisson case

Landau damping : periodic boundary conditions on $\mathbf{\Omega_x} = [0; 4\pi] \times [0; 1]$

$$f_0(\mathbf{x}, \mathbf{v}) = \frac{1}{2\pi v_{th}^2} (1 + \eta \cos(\mathbf{k} \cdot \mathbf{x})) \exp\left(-\frac{v_1^2 + v_2^2}{2 v_{th}^2}\right),$$

with $k_1 = 0.5,\ k_2 = 0,\ v_{th} = 0.1,\ \eta = 0.1.$





Conclusion

- numerical scheme to solve highly oscillatory ODEs – need to know the (numerical) fast period.
- allows big time steps with respect to this period.

Outlook

- long time behaviour of the scheme
- numerical comparisons with the two-scale limit model and the guiding center model
- Finite Larmor Radius test case

THANK YOU !

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