Semi-Lagrangian Methods Based on Cartesian Mesh for Plasma Turbulence

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Outline

The evolution of the density of particles $f(t, \mathbf{x}, \mathbf{v})$ in the phase space $(\mathbf{x}, \mathbf{v}) \in \mathbb{R}^3 \times \mathbb{R}^3$, can be described by the Vlasov equation,

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + F(t, \mathbf{x}, \mathbf{v}) \cdot \nabla_{\mathbf{v}} f = 0, \tag{1}$$

where the force field $F(t, \mathbf{x}, \mathbf{v})$ is coupled with the distribution function f giving a non linear system.
Vlasov Equation (1) has form

$$\frac{\partial f}{\partial t} + \mathbf{A} \cdot \nabla f = 0, \tag{2}$$

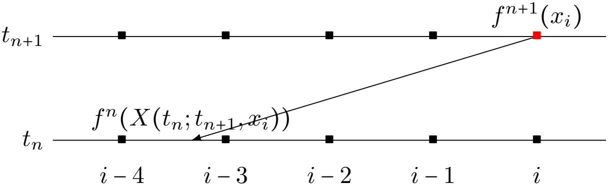
where $f : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$ and $\mathbf{A} : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$. For given $s \in \mathbb{R}^+$, the differential system

$$\begin{cases} \frac{d\mathbf{X}}{dt} = \mathbf{A}(t, \mathbf{X}), \\ \mathbf{X}(s) = \mathbf{x}, \end{cases}$$

is associated with the transport equation (2). We denote its solution by $\mathbf{X}(t; s, \mathbf{x})$.

The classical semi-Lagrangian method is decomposed into two steps for computing f^{n+1} from f^n : (cf. : Sonnendrücker *et al.*, Filbet *et al.*)

1. For each mesh point $\mathbf{x}_i$ of phase space, compute $\mathbf{X}(t_n; t_{n+1}, \mathbf{x}_i)$.
2. We obtain the value of $f^{n+1}(\mathbf{x}_i)$ by computing $f^n(\mathbf{X}(t_n; t_{n+1}, \mathbf{x}_i))$ by interpolation.

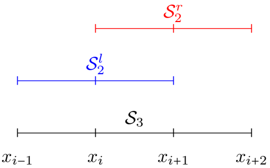


Interpolation methods :

- Cubic spline interpolation : the most used, but has spurious oscillations, high communication
- Lagrange interpolation P_3
- Hermite interpolation H_3

Lagrange WENO Interpolation $\tilde{P}_3$

To construct $f^n(x)$ in $[x_i, x_{i+1}]$, we define firstly Lagrange polynomials p_l, p_r, p_3, which verify :



$$p_r(x_i)=f_i,\quad p_r(x_{i+1})=f_{i+1},\quad p_r(x_{i+2})=f_{i+2},$$

$$p_l(x_{i-1})=f_{i-1},\quad p_l(x_i)=f_i,\quad p_l(x_{i+1})=f_{i+1},$$

$$P_3(x_{i-1})=f_{i-1},\quad P_3(x_i)=f_i,\quad P_3(x_{i+1})=f_{i+1},\quad P_3(x_{i+2})=f_{i+2}.$$

Then Lagrange WENO (LWENO) interpolation is written as

$$\tilde{P}_3(x) = w_l(x)\, p_l(x) + w_r(x)\, p_r(x).$$

where w_l and w_r are WENO weights such that :

- In the case "f is smooth" in $\mathcal{S}_3$,

$$w_l(x) \approx c_l(x) = \frac{x_{i+2} - x}{3\Delta x},\qquad w_r(x) \approx c_r(x) = \frac{x - x_{i-1}}{3\Delta x}.$$

- In the case "f is smooth" in $\mathcal{S}_2^l$ or $\mathcal{S}_2^r$,

$$w_l(x) \approx 1, w_r(x) \approx 0 \quad \text{or} \quad w_l(x) \approx 0, w_r(x) \approx 1.$$

Computation of Weights and Smoothness Indicators

To measure the smoothness, we introduce smoothness indicators

$$\beta_l = \int_{x_i}^{x_{i+1}} \Delta x (p_l')^2 + \Delta x^3 (p_l'')^2 dx = \frac{13}{12}(f_{i-1} - 2f_i + f_{i+1})^2 + (f_{i+1} - f_i)^2,$$

$$\beta_r = \int_{x_i}^{x_{i+1}} \Delta x (p_r')^2 + \Delta x^3 (p_r'')^2 dx = \frac{13}{12}(f_i - 2f_{i+1} + f_{i+2})^2 + (f_{i+1} - f_i)^2.$$

WENO weights are given by :

$$w_l = \frac{\alpha_l}{\alpha_l + \alpha_r}, \quad w_r = 1 - w_l,$$

where

$$\alpha_l = \frac{c_l}{(\varepsilon + \beta_l)^2}, \quad \alpha_r = \frac{c_r}{(\varepsilon + \beta_r)^2}.$$

For a fixed $x_p \in [x_i, x_{i+1}]$,

1. if "f is smooth" in the stencil $\mathcal{S}_3$, then $f(x_p) - \tilde{P}_3(x_p) = \mathcal{O}(\Delta x^4)$;
2. if "f is at least smooth" in one of stencils $\mathcal{S}_2^l$ or $\mathcal{S}_2^r$, then

$$f(x_p) - \tilde{P}_3(x_p) = \mathcal{O}(\Delta x^3).$$

To construct $\tilde{H}_3$ in $[x_i, x_{i+1}]$, we define

$$H_3(x_i) = f_i, \quad H_3'(x_i) = f_i', \quad H_3(x_{i+1}) = f_{i+1}, \quad H_3'(x_{i+1}) = f_{i+1}',$$

$$h_l(x_i) = f_i, \quad h_l(x_{i+1}) = f_{i+1}, \quad h_l'(x_i) = f_i',$$

$$h_r(x_i) = f_i, \quad h_r(x_{i+1}) = f_{i+1}, \quad h_r'(x_{i+1}) = f_{i+1}'.$$

Assuming f_i' are known, we introduce Hermite WENO (HWENO1) interpolation

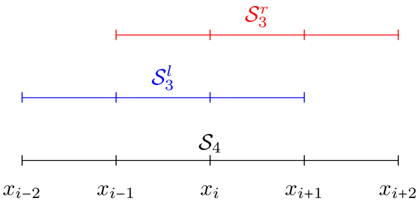
$$\tilde{H}_3(x) = w_l(x)\, h_l(x) + w_r(x)\, h_r(x),$$

where w_l and w_r are computed as previously, but with a new smoothness indicators β_l, β_r and convex coefficients c_l, c_r

$$\beta_l = (f_i - f_{i+1})^2 + \frac{13}{3}((f_{i+1} - f_i) - \Delta x f_i')^2, \quad c_l = \frac{x_{i+1} - x}{\Delta x},$$

$$\beta_r = (f_i - f_{i+1})^2 + \frac{13}{3}((f_{i+1} - f_i) - \Delta x f_{i+1}')^2, \quad c_r = \frac{x - x_i}{\Delta x}.$$

We use LWENO interpolation to compute first derivatives $\tilde{f}_i'$.



$$f_r'(x_i) = \frac{1}{6\Delta x}(-2f_{i-1} - 3f_i + 6f_{i+1} - f_{i+2}),$$

$$f_l'(x_i) = \frac{1}{6\Delta x}(f_{i-2} - 6f_{i-1} + 3f_i + 2f_{i+1}),$$

$$f_i' = \frac{1}{12\Delta x}(8(f_{i+1} - f_{i-1}) - (f_{i+2} - f_{i-2})).$$

We reduce that

$$f_i' = \frac{1}{2} f_l'(x_i) + \frac{1}{2} f_r'(x_i).$$

Thus optimized first derivative is

$$\tilde{f}_i' = w_l(x_i)\, f_l'(x_i) + w_r(x_i)\, f_r'(x_i),$$

where $w_l(x_i)$, $w_r(x_i)$ are LWENO type weights.

Modified Hermite WENO Interpolation

We propose a slightly modified Hermite WENO (HWENO2) interpolation.
In the interval $[x_i, x_{i+1}]$, we modify $\tilde{f}'_i$ and $\tilde{f}'_{i+1}$ as follows :

- If $f'_l(x_i) \cdot f'_r(x_i) \leq 0$ or $f'_l(x_{i+1}) \cdot f'_r(x_{i+1}) \leq 0$

$$\tilde{f}'_i = \tilde{f}'_{i+1} = \frac{f_{i+1} - f_i}{\Delta x},$$

- Otherwise

$$\begin{cases} \tilde{f}'_i & = & w_l(x_i) f'_l(x_i) + w_r(x_i) f'_r(x_i), \\ \tilde{f}'_{i+1} & = & w_l(x_{i+1}) f'_l(x_{i+1}) + w_r(x_{i+1}) f'_r(x_{i+1}). \end{cases}$$

Properties of HWENO2 interpolation :

1. For Δx sufficiently small, HWENO2 interpolation has the same precision as HWENO1 interpolation if “f is smooth”;
2. HWENO2 interpolation is less oscillating than HWENO1 interpolation.

We consider 1D transport equation

$$\partial_t f + v\partial_x f = 0, \quad x \in [0,1], \quad t \geq 0.$$

The periodic boundary condition is used.

- Computational time for $n_x = 1024$

	Spline	Lagrange	LWENO	Hermite	HWENO1	HWENO2
Time	1.87	1.65	1.65	1.66	1.68	1.68

- Error between exact solution and approximated solution for smooth solution case

n_x	128		256		512		1024	
	$\|\cdot\|_1$	r	$\|\cdot\|_1$	r	$\|\cdot\|_1$	r	$\|\cdot\|_1$	r
Spline	1.28e-7	2.94	1.60e-8	3.00	2.00e-9	3.00	2.50e-10	3.00
Lagrange	1.53e-6	2.94	1.92e-7	3.00	2.39e-8	3.00	2.99e-9	3.00
LWENO	1.55e-6	2.99	1.92e-7	3.01	2.40e-8	3.00	2.99e-9	3.00
Hermite	1.30e-7	2.99	1.61e-8	3.01	2.00e-9	3.01	2.50e-10	3.00
HWENO1	1.31e-7	3.04	1.61e-8	3.00	2.00e-9	3.00	2.50e-10	3.00
HWENO2	1.31e-7	3.04	1.61e-8	3.00	2.00e-9	3.00	2.50e-10	3.00

1D Test

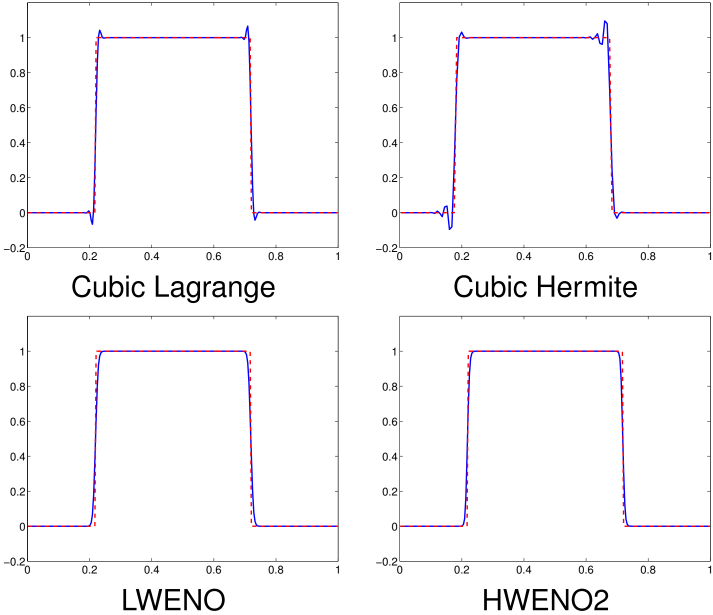
- Error between exact solution and approximated solution for discontinuous solution case

n_x	128		256		512		1024	
	$\|\cdot\|_1$	r	$\|\cdot\|_1$	r	$\|\cdot\|_1$	r	$\|\cdot\|_1$	r
Spline	1.25e-2	0.89	6.69e-3	0.91	3.56e-3	0.91	2.11e-3	0.75
Lagrange	1.43e-2	0.65	8.78e-3	0.70	5.13e-3	0.77	3.14e-3	0.71
LWENO	1.72e-2	0.67	1.07e-2	0.67	6.47e-3	0.72	3.86e-3	0.75
Hermite	1.37e-2	0.90	7.78e-3	0.82	4.65e-3	0.74	2.74e-3	0.76
HWENO1	1.60e-2	0.74	9.70e-3	0.72	5.76e-3	0.75	3.38e-3	0.77
HWENO2	1.63e-2	0.75	9.61e-3	0.77	5.57e-3	0.78	3.20e-3	0.80

- Total variation of discontinuous solution wrt exact total variation

n_x	128	256	512	1024
Spline	9.75e-1	8.25e-1	9.08e-1	7.90e-1
Lagrange	4.90e-1	4.65e-1	4.65e-1	4.83e-1
LWENO	2.87e-5	6.77e-5	1.22e-4	1.87e-4
Hermite	8.93e-1	9.87e-1	9.25e-1	9.75e-1
HWENO1	7.74e-4	1.41e-3	2.05e-3	2.73e-3
HWENO2	-4.44e-16	8.88e-16	0	0

We take $n_x = 128$, cfl=10.



From numerical evidence, we observe :

- 3rd order method in space for smooth solution.
- Control of spurious oscillations, *i.e.* control of TV.
- Preserve positivity, *i.e.*

$$f_0 \geq 0 \quad \Rightarrow \quad f(t) \geq 0.$$

- Control maximum, *i.e.*

$$\|f(t)\|_\infty \leq \|f_0\|_\infty.$$

- Mass conservation is observed for free transport.

Prove the above properties of our schemes.

The Guiding-Center model has been derived to describe highly magnetized plasma in the transverse plane of a Tokamak.

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \mathbf{U} \cdot \nabla_{\mathbf{x}_\perp} \rho = 0, \\ \mathbf{U} = \mathbf{E}^\perp, \\ -\Delta_{\mathbf{x}_\perp} \phi = \rho. \end{cases}$$

Boundary condition :

$$\phi(\mathbf{x}_\perp) = 0, \quad \mathbf{x}_\perp \in \partial D,$$

where ∂D can be arbitrary boundary.

If f is smooth, we have

(1) Maximum principle : $0 \leq \rho(t, \mathbf{x}_\perp) \leq \max_{\mathbf{x}_\perp \in D}(\rho(0, \mathbf{x}_\perp))$.

(2) L^p norm conservation : $\frac{d}{dt}\left(\int_D (\rho(t, \mathbf{x}_\perp))^p d\mathbf{x}_\perp\right) = 0$.

(3) Energy conservation : $\frac{d}{dt}\left(\int_D |\nabla \phi|^2 d\mathbf{x}_\perp\right) = 0$.

We use the semi-Lagrangian method for solving the transport equation.

1. To find the characteristic foot is equivalent to solve

$$\begin{cases} \frac{d}{dt}\mathbf{X} = \mathbf{U}(t, \mathbf{X}), \\ \mathbf{X}(s) = \mathbf{x}_\perp. \end{cases}$$

   This system can be solved by using the parabolic assumption :
   A second order scheme reads

$$\frac{\mathbf{x}_\perp - \mathbf{X}(t^{n-1}, t^{n+1}, \mathbf{x}_\perp)}{2\Delta t} = \mathbf{U}(\mathbf{X}(t^n, t^{n+1}, \mathbf{x}_\perp), t_n).$$

   Assuming that $\mathbf{U}$ is constant between t_{n+1} and t_{n-1}, we get

$$\mathbf{d} = \Delta t \mathbf{U}(\mathbf{x}_\perp - \mathbf{d}, t_n),$$

   where $\mathbf{d}$ is the shift vector in $\mathbf{x}_\perp$ plane.
   This equation can be solved by a Taylor method.
2. HWENO2 interpolation method is used.
   2D interpolation can be proceeded dimension by dimension.

Discretization of Poisson Equation

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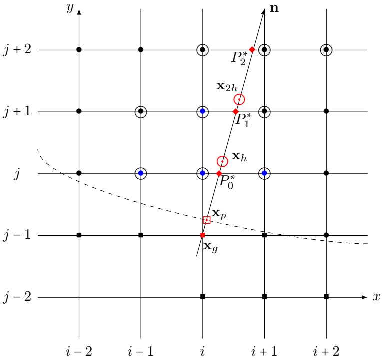
We discretize Poisson equation in an arbitrary domain.



• is interior point, ▪ is ghost point, ⊡ is the point at the boundary, ◯ is the point for interpolation, the dashed line is the boundary.

- Classical five points finite difference scheme is used.
- Extrapolation technique for treatment of B.C.
  We extrapolate $\phi_{i,j-1}$ on the normal direction $\mathbf{n}$

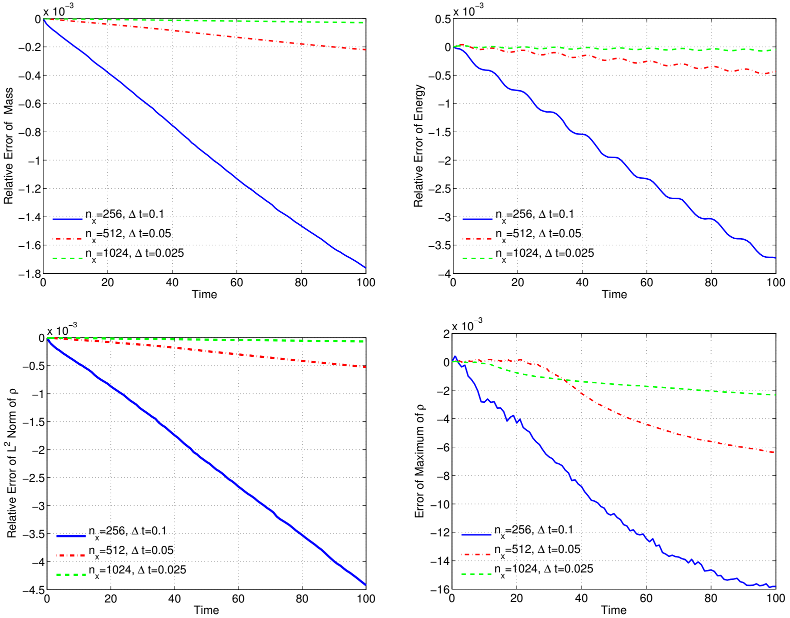
$$\phi_{i,j-1} = \tilde{w}_p\phi(\mathbf{x}_p) + \tilde{w}_h\phi(\mathbf{x}_h) + \tilde{w}_{2h}\phi(\mathbf{x}_{2h}),$$

where

- $\phi(\mathbf{x}_p)$ is known by Dirichlet B.C.
- $\phi(\mathbf{x}_h)$, $\phi(\mathbf{x}_{2h})$ are determined by interpolation.

Therefore, $\phi_{i,j-1}$ is approximated from the interior domain.

For a smooth solution, we have



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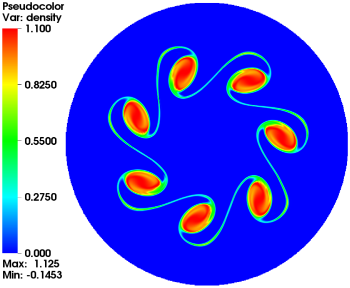
We now consider the diocotron instability for an annular electron layer. The initial data is given by (cf. : Pétri, Mehrenberger)

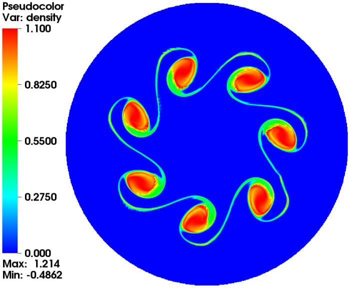
$$\rho_0(\mathbf{x}_\perp) = \begin{cases} 1 + \varepsilon \cos(\ell\theta), & \text{if } r^- \leq \sqrt{x^2+y^2} \leq r^+, \\ 0, & \text{otherwise,} \end{cases}$$

where $\ell = 7$, $\varepsilon = 0.1$.

Let us consider a disk domain $D = \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} \leq R\}$. cfl $\approx$ 10.

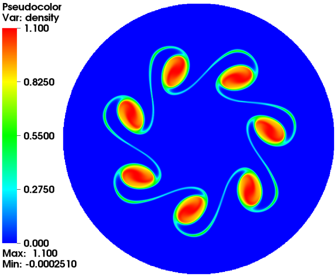


Cubic Lagrange

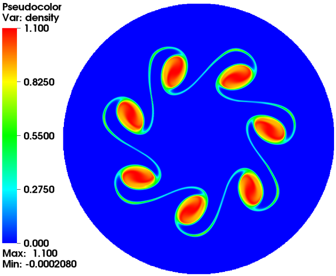


Cubic Hermite

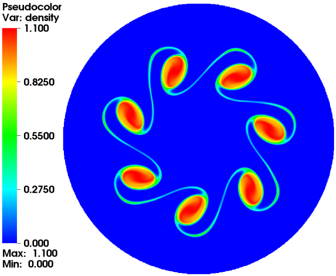
Diocotron Instability Simulation



LWENO

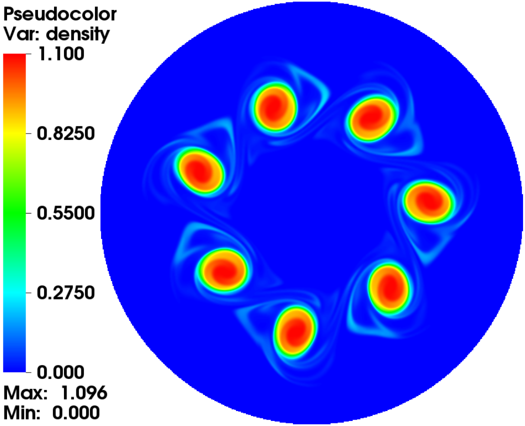


HWENO1



HWENO2

Evolution of Diocotron instability simulation

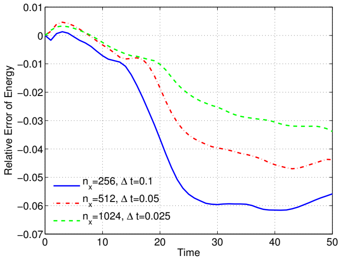


Note that the semi-Lagrangian method does not preserve mass. We thus add a least square method, denoted by LS, to enforce the mass conservation.

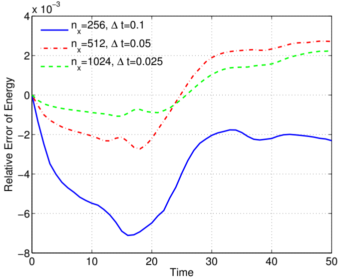
Suppose that the density ρ^{n+1} is obtained by the semi-Lagrangian method, then in this particular case the LS procedure reads

$$\mathrm{LS}(\rho) = \frac{M^n}{M^{n+1}}\rho^{n+1},$$

where $M^n = \int_{\Omega_{\mathbf{x}_\perp}} \rho^n d\mathbf{x}_\perp$.

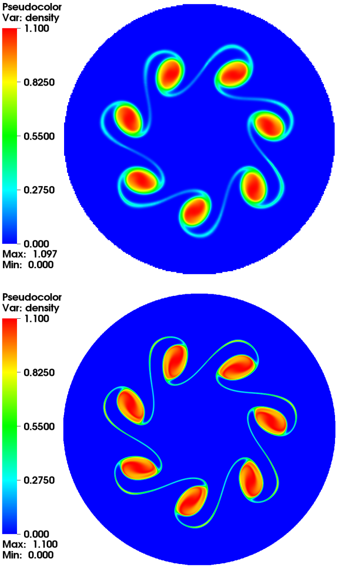


Without LS procedure

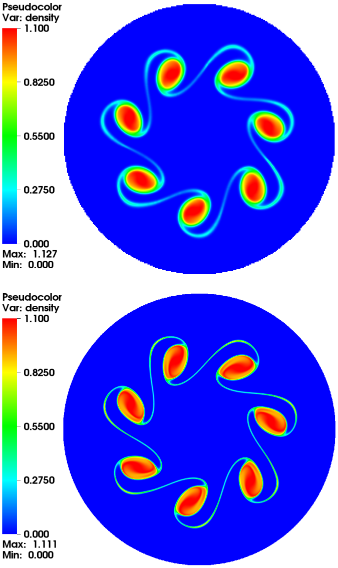


With LS procedure

Enforce Mass Conservation



Without LS procedure



With LS procedure

Normalized Drift-Kinetic model reads (cf. Grandgirard *et al.*)

$$\begin{cases} \frac{\partial f}{\partial t} + \mathbf{U} \cdot \nabla_{\mathbf{x}_\perp} f + v_\parallel \partial_z f + E_\parallel \partial_{v_\parallel} f = 0, \\ \mathbf{U} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}, \\ -\nabla_\perp \cdot \left(\frac{\rho_0(\mathbf{x}_\perp)}{B} \nabla_\perp \phi \right) + \frac{\rho_0(\mathbf{x}_\perp)}{T_e(\mathbf{x}_\perp)} (\phi - \bar{\phi}) = \rho - \rho_0. \end{cases}$$

In the following simulation, we consider a cylinder domain

$$\Omega_{\mathbf{x}} = \{(x, y, z) \in \mathbb{R}^3 : \sqrt{x^2 + y^2} \leq R, 0 \leq z \leq L_z\}.$$

Boundary condition :

- $\phi(\mathbf{x}) = 0$ on $\partial D \times [0, L_z]$, where $\partial D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = R\}$.
- Periodic boundary condition in z-direction.

- The Drift-Kinetic Vlasov equation can be split into three equations

$$\begin{cases} \dfrac{\partial f}{\partial t} + \mathbf{U} \cdot \nabla_{\mathbf{x}_\perp} f = 0, \\ \dfrac{\partial f}{\partial t} + v_\parallel \partial_z f = 0, \\ \dfrac{\partial f}{\partial t} + E_\parallel \partial_{v_\parallel} f = 0. \end{cases}$$

The Strang splitting method can be used for time discretization.

- Averaging the the quasi-neutrality equation in z-direction, we get a 2D average equation ($\bar{\phi} = \int_0^{L_z} \phi dz$) :

$$-\nabla_\perp \cdot \left(\frac{\rho_0(\mathbf{x}_\perp)}{B} \nabla_\perp \bar{\phi} \right) = \bar{\rho} - \rho_0.$$

Taking difference between the quasi-neutrality equation and the average equation, it yields a fluctuation equation ($\phi' = \phi - \bar{\phi}$):

$$-\nabla_\perp \cdot \left(\frac{\rho_0(\mathbf{x}_\perp)}{B} \nabla_\perp \phi' \right) + \frac{\rho_0(\mathbf{x}_\perp)}{T_e(\mathbf{x}_\perp)} \phi' = \rho' - \bar{\rho}.$$

The fluctuation equation can be solve slice by slice in z-direction.

The plasma is initialized by exciting a single or random ion temperature gradient (ITG) model (m, n) (where m is a poloidal mode and n is a toroidal mode) :

$$f = f_{\mathrm{eq}} + \delta f.$$

The equilibrium part f_{eq} is chosen as a local Maxwellian

$$f_{\mathrm{eq}}(r, v_{\|}) = \frac{n_0(r)}{(2\pi T_i(r))^{1/2}} \exp\left(-\frac{v_{\|}^2}{2T_i(r)}\right),$$

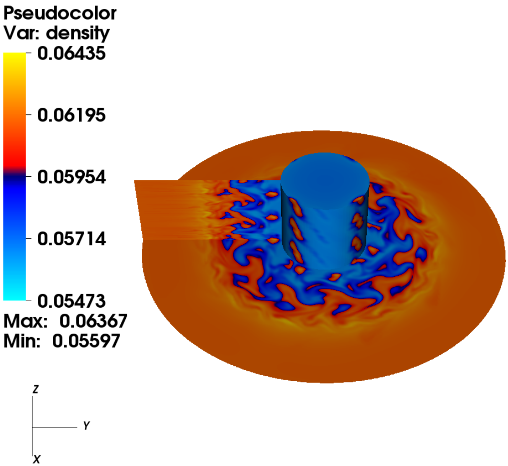
while the perturbation δf is determined as

$$\delta f = f_{\mathrm{eq}} g(r) h(v_{\|}) \delta p(z, \theta),$$

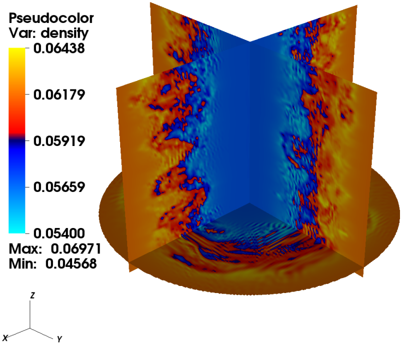
where

$$g(r = 0) \sim g(r = r_{\max}) \sim 0,$$
$$h(v_{\|} = v_{\|\,\min}) \sim h(v_{\|} = v_{\|\,\max}) \sim 0,$$
$$\delta p(z,\theta) = \varepsilon \cos\left(\frac{2\pi n}{L_z} z + m\theta\right), \quad \delta p(z,\theta) = \sum_{n,m} \varepsilon_{n,m} \cos\left(\frac{2\pi n}{L_z} z + m\theta + \phi_{n,m}\right).$$

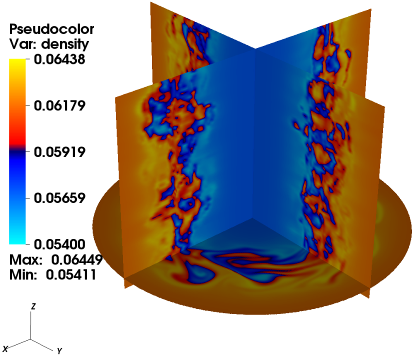
Evolution of ion turbulence simulation (single mode, cfl $\approx$ 5)



Comparison of different interpolation methods (random mode)



Cubic Hermite



HWENO2

Conclusion

- Classical semi-Lagrangian method for Vlasov equation
  - ⋆ Parabolic assumption method for search characteristic
  - ⋆ WENO type method for interpolation
  - ⋆ HWENO2 interpolation method : quasi non-oscillating, high accuracy, low communication
- Plasma turbulence simulation based on Cartesian mesh
  - ⋆ Extrapolation technique for Dirichlet B.C. of elliptic equation
  - ⋆ 2D Diocotron instability simulation
  - ⋆ 4D ion turbulence simulation
  - ⋆ Advantages :
    - Non singularity as in polar coordinates
    - Easier to adapt arbitrary domain

Perspectives

- Prove the properties of semi-Lagrangian method for Vlasov equation
- Improve conservation laws of semi-Lagrangian method for Vlasov equation