Asymptotic preserving schemes (AP) for magnetically confined plasmas

Highly anisotropic elliptic/parabolic equations

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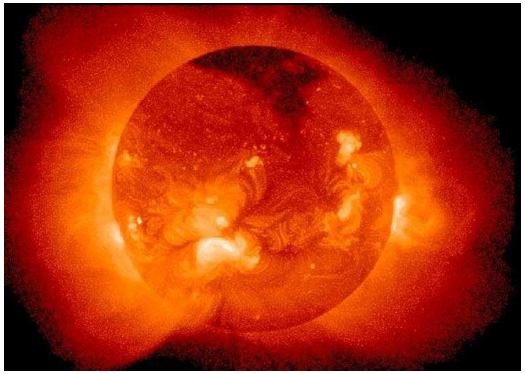
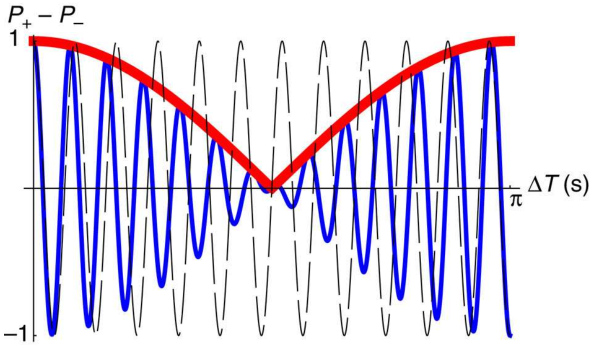
Introduction/Motivation

Objective: Numerical study of highly anisotropic, multi-scale problems

- Many pb. in nature exhibit multi-scale behaviours, which can be rather different in character
- Typical: occurence of one or several small/large parameters (Reynolds, Peclet, Mach nbr. etc)
- General, unified treatment is impossible



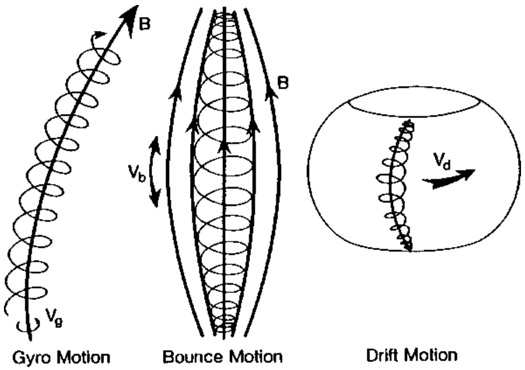


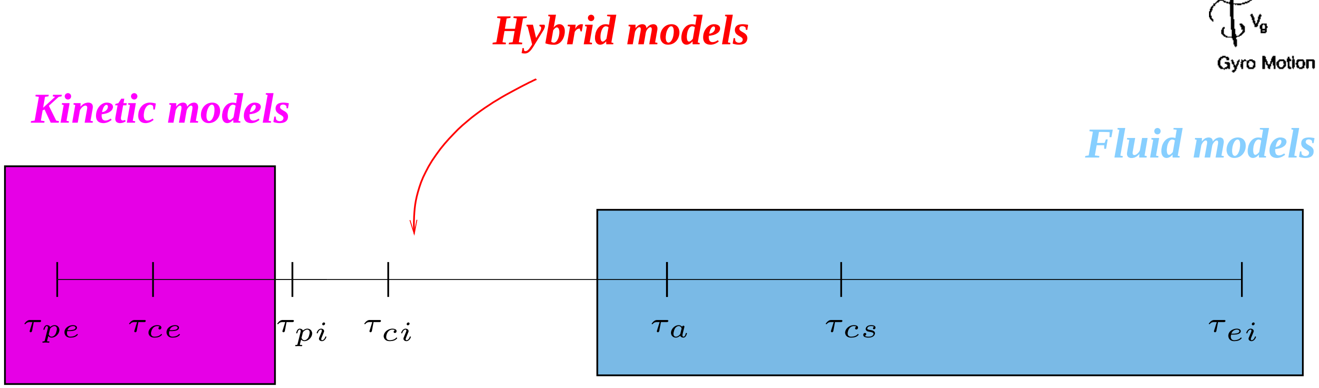
Multi-scale plasma dynamics

Plasma dynamics is characterized by multi-scale phenomena

$\rightarrow$ Strong magn. fields create anisotropies

$\rightarrow$ Particles gyrate around the field lines





$\tau_{pe,pi}$: Inv. electr./ion plasma freq.

$\tau_{ce,ci}$: Electr./ion cyclotron period

τ_a: Alfen wave period

τ_{cs}: Ion sound period

τ_{ei}: Electr-ion collision time

λ_D: Debye length

$\rho_{e,i}$: Electr./ion Larmor radius

$\delta_{e,i} = c/\omega_{pe,pi}$: Electr./ion skin depth

$\omega_{pe,pi}$: Electr./ion plasma frequency

c: sound speed

Multi-scale problems

A small-scale numerical simulation is out of reach

- requires mesh-sizes dependent on small scale param. $\varepsilon \ll 1$
- excessive computational time and memory space are needed to capture small scales

It is not always of interest to resolve the details at the small scale. Multi-scale strategies are much more adequate!

- homogeneisation, domain decomposition, multi-grids, multi-scale methods based on wavelets or finite elements, multi-scale variational methods

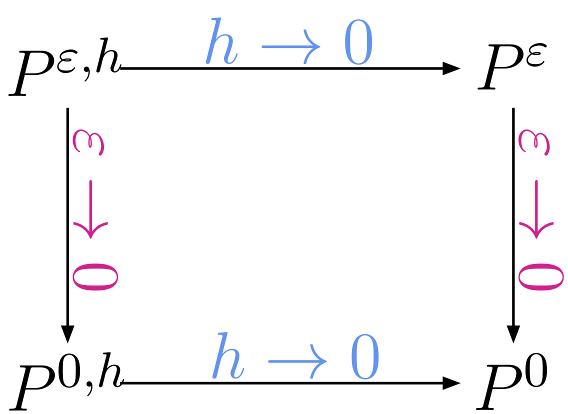
Essential feature of these methods

- capture efficiently the large scale behavior of the solution, without resolving the small scale features

Difficulty: Resolution of multiscale pb. can be very difficult, if the pb. becomes singular, as one of the parameters $\varepsilon \to 0$

- (P^ε) sing. perturbed pb. with sol. f_ε
- the seq. f_ε converges towards f_0, sol. of a limit pb. (P^0)
- the limit pb. (P^0) is different in type from the initial (P^ε)
- standard schemes would require $\Delta t, \Delta x \sim \varepsilon$ for stability

Definition: A scheme $P^{\varepsilon,h}$ is AP iff it is convergent for $h \to 0$ uniformely in ε, *i.e.*



AP-procedure:

- requires that the limit problem (P^0) is identified and well-posed
- consists in trying to mimic at discrete level the asymptotic behaviour of the sing. perturbed pb. sol. f_ε
- requires a sufficient degree of implicitness (not obvious)

Advantages:

- gives accurate and stable results, with no restrictions on the computational mesh
- enables to capture automatically the Limit model P^0, if $\varepsilon \to 0$ (micro-macro transition)
- no more coupling needed, if $\varepsilon(x)$ is variable

Kinetic models and specific limit regimes

- Fundamental kinetic model: Vlasov/Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f + \frac{q}{m}(E + v \times B) \cdot \nabla_v f = Q(f)$$

Several small scales/parameters occur, leading to diff. regimes:

- Hydrodynamic scaling [Filbet/Jin; Dimarco/Pareschi]

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} Q(f)$$

➠ $0 < \varepsilon \ll 1$: mean free path (Knudsen nbr.)

➠ in the limit $\varepsilon \to 0$, one gets the compressible Euler eq.

➠ AP-scheme: Decomposition of the source term in stiff- and non-stiff part

$$\frac{Q(f)}{\varepsilon} = \frac{Q(f) - P(f)}{\varepsilon} + \frac{P(f)}{\varepsilon}$$

Kinetic models and specific limit regimes

- Drift-Diffusion scaling [Klar; Lemou/Mieussens]

$$\partial_t f + \frac{1}{\varepsilon}(v \cdot \nabla_x f + \nabla_x \Phi \cdot \nabla_v f) + \frac{1}{\varepsilon^2} Q(f) = G$$

- $0 < \varepsilon \ll 1$: mean free path
- in the limit $\varepsilon \to 0$, one gets the Drift-Diffusion model
- AP-scheme: Micro-Macro decomp. $f = \rho M + \varepsilon g$

Vlasov-Poisson quasi-neutral limit [Belaouar;Crouseilles;Degond;Deluzet;Sonnendrucker;Navoret;Vignal]

$$\begin{cases} \partial_t f + v \partial_x f + \partial_x \Phi \partial_v f = 0 \quad \text{or} \quad = \frac{1}{\varepsilon} Q(f) \\ -\lambda^2 \partial_{xx} \Phi = 1 - \rho \end{cases}$$

- $0 < \lambda \ll 1$: rescaled Debye length; $0 < \varepsilon \ll 1$: mean free path
- AP-scheme: Reformulation of the Poisson equation

Kinetic models and specific limit regimes

- Vlasov-Maxwell quasi-neutral limit [Degond/Deluzet/Dimarco/Doyen]

- High-field limit, strong magn. fields [Bostan, Frenod, Golse, Saint-Raymond]

$$\partial_t f + v_{||} \cdot \nabla_x f + E \cdot \nabla_v f + \frac{1}{\tau} v_\perp \cdot \nabla_x f + \frac{1}{\varepsilon}(v \times B) \cdot \nabla_v f = 0$$

➠ $0 < \varepsilon \ll 1$: cycl. period; $0 < \tau \ll 1$: Larmor radius

➠ in the limit $\varepsilon, \tau \to 0$, one gets the finite Larmor radius approx. or the guiding-center approx.

➠ asymptotical analysis:
- Study of the dominant operator $\mathcal{T} := (v(p) \times B) \cdot \nabla_v$
- Projection of the eq. on $\ker \mathcal{T}$= averaging along the charact. flow associated to $\mathcal{T}$

➠ construction of AP-scheme mimics this asymp. analysis

- Euler-Poisson quasi-neutral limit [Crispel/Degond/Vignal]

$$\begin{cases} \partial_t n + \nabla \cdot (n\, u) = 0 \\ \partial_t (n\, u) + \nabla \cdot (n\, u \otimes u) + \nabla p(n) = n\, \nabla \Phi \\ -\lambda^2 \Delta \Phi = 1 - n \end{cases}$$

➠ $0 < \lambda \ll 1$: rescaled Debye length

- High-field limit, Euler-Lorentz [Brull;Degond;Deluzet;Mouton;Sangam;Vignal]

$$\begin{cases} \partial_t n + \nabla \cdot (n\, u) = 0 \\ \partial_t (n\, u) + \nabla \cdot (n\, u \otimes u) + \dfrac{1}{\tau} \nabla p(n) = \dfrac{1}{\tau} n\, (E + u \times B) \end{cases}$$

➠ $0 < \tau \ll 1$: rescaled gyro-period

Fluid models and specific limit regimes

- Low Mach-nbr. limit [Degond/Tang; Cordier/Degond/Kumbaro]

$$\begin{cases} \partial_t n + \nabla \cdot (n\, u) = 0 \\ \partial_t (n\, u) + \nabla \cdot (n\, u \otimes u) + \dfrac{1}{\varepsilon^2} \nabla p(n) = 0 \end{cases}$$

  - ➠ $0 < \varepsilon \ll 1$: rescaled Mach-nbr.
  - ➠ in the limit $\varepsilon \to 0$, one gets the incompressible Euler eq.
  - ➠ AP-scheme: Stiff term is decomposed as

$$\frac{1}{\varepsilon^2} \nabla p(n) = \alpha \nabla p(n) + \frac{1 - \alpha \varepsilon^2}{\varepsilon^2} \nabla p(n)$$

- Highly anisotropic potential/temp. eq. [Deluzet/Lozinski/Mentrelli/Narski/Negulescu]

$$-\frac{1}{\varepsilon} \nabla_{||} \cdot (A_{||} \nabla_{||} \phi) - \nabla_{\perp} \cdot (A_{\perp} \nabla_{\perp} \phi) = f$$

$$\partial_t T - \frac{1}{\varepsilon} \nabla_{||} \cdot (K_{||} \nabla_{||} T) - \nabla_{\perp} \cdot (K_{\perp} \nabla_{\perp} T) = 0$$

Anisotropic elliptic equation

Work based on:

[1] P. Degond, F. Deluzet, C. Negulescu "An asymptotic preserving scheme for strongly anisotropic elliptic problems", SIAM-MMS.
[2] P. Degond, F. Deluzet, A. Lozinski, J. Narski, C. Negulescu "Duality based Asymptotic-Preserving Method for highly anisotropic diffusion equations", CMS.
[3] P. Degond, A. Lozinski, J. Narski, C. Negulescu "An Asymptotic-Preserving method for highly anisotropic elliptic equations based on a micro-macro decomposition", JCP.

Macroscopic nature of magnetically confined plasma dynamics can be described via the one-fluid (MHD) model

- Continuity equation of plasma charge

$$\partial_t \rho_c + \mathrm{div} \cdot j = 0\,,$$

$$\rho_c := q n_i - e n_e\,, \quad j = q n_i u_i - e n_e u_e\,, \quad q = eZ$$

- Quasi-neutrality ($|n_e - Z n_i| \ll n_e$) implies

$$\rho_c \approx 0 \Rightarrow \nabla \cdot j = 0$$

- Ohm's law

$$j = \sigma(E + v \times B)\,, \quad v = \frac{n_e m_e u_e + n_i m_i u_i}{n_e m_e + n_i m_i}\,, \quad E = -\nabla \phi$$

Theme: Numerical resolution of highly anisotropic elliptic eq.

$$\begin{cases} -\nabla \cdot (\mathbb{A} \nabla \phi) = f \,, \quad \text{on } \Omega \\ \phi = 0 \quad \text{on } \partial\Omega_D \,, \quad \partial_z \phi = 0 \quad \text{on } \partial\Omega_z \,, \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ or $\Omega \subset \mathbb{R}^3$ and $\partial\Omega = \partial\Omega_D \cup \partial\Omega_z$.

- Diffusion matrix $\mathbb{A}$ is given by

$$\mathbb{A} = \begin{pmatrix} A_\perp & 0 \\ 0 & \frac{1}{\varepsilon} A_z \end{pmatrix} , \quad -\frac{\partial}{\partial x}\left(A_\perp \frac{\partial \phi}{\partial x}\right) - \frac{1}{\varepsilon}\frac{\partial}{\partial z}\left(A_z \frac{\partial \phi}{\partial z}\right) = f \,,$$

- $A_\perp$, A_z of same order of magnitude, bounded from below/above
- $0 < \varepsilon \ll 1$ very small
- anisotropy aligned with the z-coordinate

The Singular Perturbation pb. (P-model)

$$(P) \quad \begin{cases} -\dfrac{\partial}{\partial x}\left(A_\perp \dfrac{\partial \phi}{\partial x}\right) - \dfrac{1}{\varepsilon}\dfrac{\partial}{\partial z}\left(A_z \dfrac{\partial \phi}{\partial z}\right) = f\,, \quad \text{on} \quad \Omega\,, \\ \dfrac{\partial \phi}{\partial z} = 0 \quad \text{on} \quad \Omega_x \times \partial\Omega_z\,, \qquad \phi = 0 \quad \text{on} \quad \partial\Omega_x \times \Omega_z\,, \end{cases}$$

Letting formally $\varepsilon \to 0$, yields the reduced problem (R-model)

$$(R) \quad \begin{cases} -\dfrac{\partial}{\partial z}\left(A_z \dfrac{\partial \phi}{\partial z}\right) = 0\,, \quad \text{on} \quad \Omega\,, \\ \dfrac{\partial \phi}{\partial z} = 0 \quad \text{on} \quad \Omega_x \times \partial\Omega_z\,, \qquad \phi = 0 \quad \text{on} \quad \partial\Omega_x \times \Omega_z\,, \end{cases}$$

$\to$ R is an ill-posed problem !

$\to$ R exhibits an infinit amount of solutions $\phi(x)$.

The Limit problem (L-model)

- Numerical burden: Discretization matrix of P-model is very ill-conditioned, $cond \sim \frac{1}{\varepsilon}$.

$\rightarrow$ standard resolution meth. for lin. syst. no more efficient for $0 < \varepsilon << 1$

- However, ϕ_ε (sol. of P-model) $\rightarrow_{\varepsilon \to 0} \overline{\phi}_0$, sol. of

$$(L) \quad \begin{cases} -\frac{\partial}{\partial x}\left(\bar{A}_\perp \frac{\partial \bar{\phi}}{\partial x}\right) = \bar{f}(x)\,, & \text{on} \quad \Omega_x\,, \\ \bar{\phi} = 0 \quad \text{on} \quad \partial\Omega_x\,, \end{cases}$$

where $\bar{f}(x) := \frac{1}{L_z}\int_0^{L_z} f(x,z)dz$ is the average along z-coord.

Identification of Limit-model:

- Suppose $\phi_\varepsilon \to \phi_0$, where $\phi_0(x)$ dep. only on x
- Integrate (P_ε) in z and pass to the limit $\varepsilon \to 0$
- Averaging in z is the proj. on the kernel of dominant op.

Transition between (P_ε) and (L) pb.

Let us denote by:

- $||$ the direction of the anisotropy (here z-direction)
- $\perp$ the perpendicular direction (here x-direction)
- the bilinear forms

$$a_{||}(\phi,\psi) \quad := \quad \int_\Omega A_{||}\nabla_{||}\phi\cdot\nabla_{||}\psi\,dx\,dz\,, \quad a_\perp(\phi,\psi) := \int_\Omega (A_\perp\nabla_\perp\phi)\cdot\nabla_\perp\psi\,dx\,dz\,.$$

How to switch from sing. perturbed pb.: find $\phi^\varepsilon \in \mathcal{V}$, sol. of

$$(P_\varepsilon)\;\; a_{||}(\phi^\varepsilon,\psi) + \varepsilon a_\perp(\phi^\varepsilon,\psi) = \varepsilon(f,\psi)\,, \quad \forall\psi\in\mathcal{V}\,,$$

to Limit model: find $\phi^0 \in \mathcal{G}$, sol. of

$$(L)\;\; a_\perp(\phi^0,\psi) = \varepsilon(f,\psi)\,, \quad \forall\psi\in\mathcal{G}\,,$$

Goal: AP-scheme which switches automatically, with no hugh num. costs

Duality-Based decomposition

- Introduction of mathematical framework

$$\mathcal{V} := \{\phi \in H^1(\Omega) \ / \ \phi_{|\partial\Omega_D} = 0\}\,, \quad (\phi,\psi)_{\mathcal{V}} := (\nabla_{||}\phi, \nabla_{||}\psi)_{L^2} + \varepsilon(\nabla_\perp\phi, \nabla_\perp\psi)_{L^2}$$

- Identification of Kernel of dominant operator

$$\mathcal{G} := \{\phi \in \mathcal{V} \mid \nabla_{||}\phi = 0\}\,, \quad (\phi,\psi)_{\mathcal{G}} := (\nabla_\perp\phi, \nabla_\perp\psi)_{L^2}\,,$$

$$\mathcal{A} := \{\phi \in \mathcal{V} \mid (\phi,\psi) = 0 \ , \ \forall \psi \in \mathcal{G}\} = \{\phi \in \mathcal{V} \mid \int_{L_z} \phi(x,z)\, dz = 0\}$$

- Definition of the orthogonal projection on the Kernel

$$P : \mathcal{V} \to \mathcal{G} \ \text{ such that } \ P\phi := \frac{1}{L_z}\int_{L_z} \phi(x,z)\, dz$$

- Definition of decomposition $\mathcal{V} = \mathcal{G} \oplus^\perp \mathcal{A}$

$$\phi^\varepsilon \in \mathcal{V} \ \Rightarrow \ \phi^\varepsilon = p^\varepsilon + q^\varepsilon\,, \quad p^\varepsilon = P\phi^\varepsilon \in \mathcal{G}\,, \quad q^\varepsilon = (I - P)\phi^\varepsilon \in \mathcal{A}$$

The Asymptotic Preserving form. (AP-form.)

- Insertion of $\phi^\varepsilon = p^\varepsilon + q^\varepsilon$ in sing. perturbed pb.: $\phi^\varepsilon \in \mathcal{V}$

$$(P_\varepsilon) \;\; a_{||}(\phi^\varepsilon, \psi) + \varepsilon a_\perp(\phi^\varepsilon, \psi) = \varepsilon(f, \psi)\,, \quad \forall \psi \in \mathcal{V}\,,$$

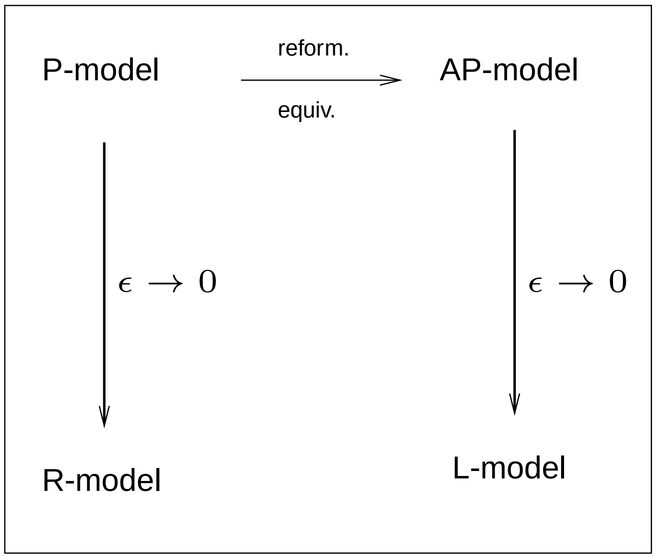
- Projection on the kernel $\Rightarrow$ Asymp.-preserv. pb.: $(p^\varepsilon, q^\varepsilon) \in \mathcal{G} \times \mathcal{A}$

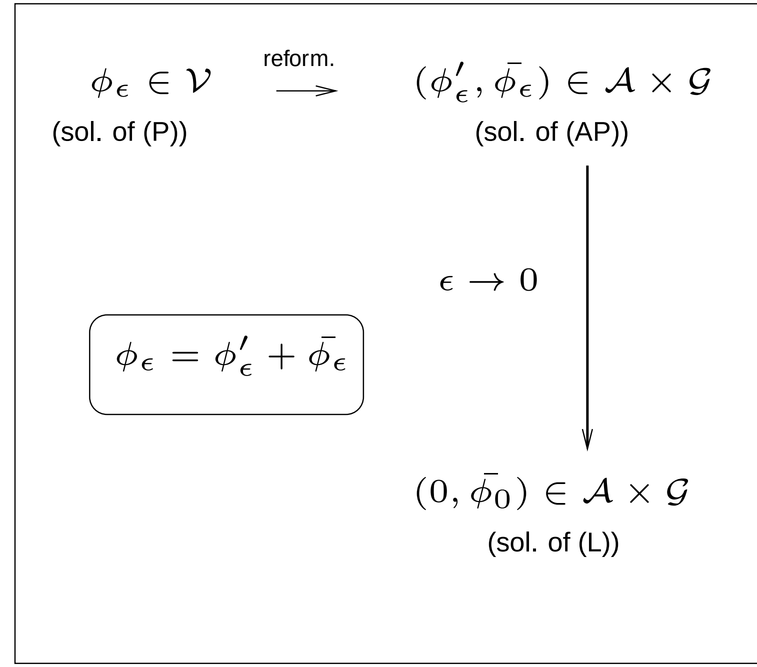
$$(AP)_\varepsilon \;\; \begin{cases} a_\perp(p^\varepsilon, \eta) + a_\perp(q^\varepsilon, \eta) = (f, \eta), & \forall \eta \in \mathcal{G}, \\ a_{||}(q^\varepsilon, \xi) + \varepsilon a_\perp(q^\varepsilon, \xi) + \varepsilon a_\perp(p^\varepsilon, \xi) = \varepsilon(f, \xi), & \forall \xi \in \mathcal{A}. \end{cases}$$

- In the limit $\varepsilon \to 0$ one gets Limit pb.: $(p^0, q^0) \in \mathcal{G} \times \mathcal{A}$

$$(L) \;\; \begin{cases} a_\perp(p^0, \eta) + a_\perp(q^0, \eta) = (f, \eta)\,, & \forall \eta \in \mathcal{G} \\ a_{||}(q^0, \xi) = 0\,, & \forall \xi \in \mathcal{A}\,, \end{cases}$$

Summary of the AP-idea





$$\mathcal{V} := \{\psi(\cdot,\cdot) \in H^1(\Omega) \;/\; \psi = 0 \text{ on } \partial\Omega_x \times \Omega_z\} \qquad \mathcal{V} = \mathcal{G} \oplus^\perp \mathcal{A}$$

$$\mathcal{G} := \{\phi \in \mathcal{V} \mid \nabla_\parallel \phi = 0\} = \{\phi(\cdot) \in H^1(\Omega_x) \;/\; \phi = 0 \text{ on } \partial\Omega_x\},$$

$$\mathcal{A} := \{\phi \in \mathcal{V} \,|(\phi,\psi) = 0 \;\;,\;\; \forall \psi \in \mathcal{G}\} = \{\phi \in \mathcal{V} \,|\, \int_{L_z} \phi(x,z)\, dz = 0\}$$

More general context (P-model)

Let b be a vector field: direction of the anisotropy (magnetic field)

$$\nabla_{||}\phi := (b\cdot\nabla\phi)b\,,\quad \nabla_\perp\phi := (Id - b\otimes b)\nabla\phi$$

$$(P_\varepsilon)\quad \begin{cases} -\frac{1}{\varepsilon}\nabla_{||}\cdot\left(A_{||}\nabla_{||}u^\varepsilon\right) - \nabla_\perp\cdot(A_\perp\nabla_\perp u^\varepsilon) = f & \text{in}\quad \Omega, \\ \frac{1}{\varepsilon}n_{||}\cdot\left(A_{||}\nabla_{||}u^\varepsilon\right) + n_\perp\cdot(A_\perp\nabla_\perp u^\varepsilon) = 0 & \text{on}\quad \Gamma_N, \\ u^\varepsilon = 0 & \text{on}\quad \Gamma_D\,. \end{cases}$$

$$\Gamma_D := \{x\in\Gamma\ /\ b(x)\cdot n(x) = 0\}\,,\quad \Gamma_N := \{x\in\Gamma\ /\ b(x)\cdot n(x)\neq 0\}\,.$$

- Introduction of mathematical framework

$$\mathcal{V} := \{u\in H^1(\Omega)\ /\ u_{|\Gamma_D} = 0\}\,,\quad (u,v)_{\mathcal{V}} := (\nabla_{||}u,\nabla_{||}v)_{L^2} + (\nabla_\perp u,\nabla_\perp v)_{L^2}$$

- Identification of Kernel of dominant operator

$$\mathcal{G} := \{u\in\mathcal{V}\ |\ \nabla_{||}u = 0\}\,,\quad (u,v)_{\mathcal{G}} := (\nabla_\perp u,\nabla_\perp v)_{L^2}\,,$$

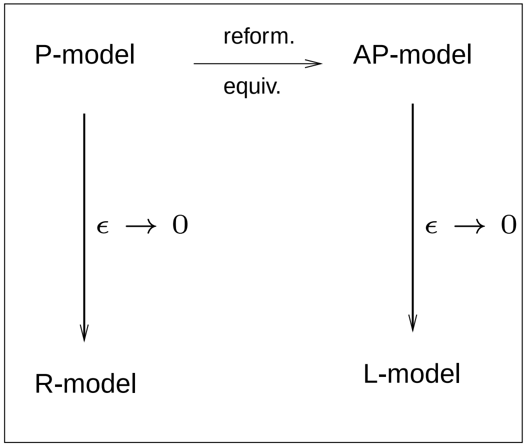
Limit problem (L-model)

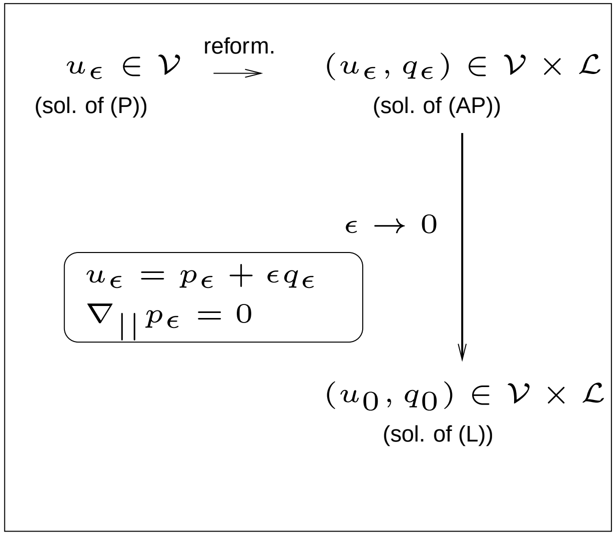
The solution u^ε of pb. $(P)_\varepsilon$

$$(P)_\varepsilon \qquad \int_\Omega A_{||}\nabla_{||}u^\varepsilon \cdot \nabla_{||}v\,dx + \varepsilon\int_\Omega (A_\perp \nabla_\perp u^\varepsilon)\cdot\nabla_\perp v\,dx = \varepsilon(f,v)\,, \quad \forall v \in \mathcal{V}\,,$$

converges for $\varepsilon \to 0$ towards u^0, sol. of

$$(L) \qquad \int_\Omega (A_\perp \nabla_\perp u^0)\cdot\nabla_\perp v\,dx = \int_\Omega f v\,dx\,, \quad \forall v \in \mathcal{G}\,.$$





Goal: AP-scheme which switches automatically between (P_ε) and (L).

Micro-Macro decomposition

- Definition of Duality-Based decomposition $\mathcal{V} = \mathcal{G} \oplus^{\perp} \mathcal{A}$

$$\phi^\varepsilon \in \mathcal{V} \;\Rightarrow\; \phi^\varepsilon = p^\varepsilon + q^\varepsilon\,, \quad p^\varepsilon = P\phi^\varepsilon \in \mathcal{G}\,, \quad q^\varepsilon = (I-P)\phi^\varepsilon \in \mathcal{A}$$

$$\mathcal{G} := \{\phi \in \mathcal{V} \mid \nabla_{||}\phi = 0\}\,, \quad \mathcal{A} := \{\phi \in \mathcal{V} \mid \int_{L_z} \phi(x,z)\,dz = 0\}$$

- Definition of the orthogonal projection on the Kernel

$$P : \mathcal{V} \to \mathcal{G} \;\text{ such that }\; P\phi := \frac{1}{L_z}\int_{L_z} \phi(x,z)\,dz$$

- New Micro-Macro decomposition (based on Hilbert-Ansatz idea)

$$u^\varepsilon = p^\varepsilon + \varepsilon q^\varepsilon$$

where

$$\nabla_{||}p^\varepsilon = 0\,, \quad \nabla_{||}u^\varepsilon = \varepsilon \nabla_{||}q^\varepsilon\,, \quad q^\varepsilon_{|\Gamma_{in}} = 0$$

Highly anisotropic elliptic problem :

$$(P_\varepsilon) \quad \int_\Omega A_\perp \nabla_\perp u^\varepsilon \cdot \nabla_\perp v\, dx + \int_\Omega \frac{A_{||}}{\varepsilon} \nabla_{||} u^\varepsilon \cdot \nabla_{||} v\, dx = \int_\Omega f v\, dx, \quad \forall v \in \mathcal{V}$$

Micro-Macro decomposition: $u^\varepsilon = p^\varepsilon + \varepsilon q^\varepsilon, \quad \mathcal{V} = \mathcal{G} \oplus \mathcal{L}$

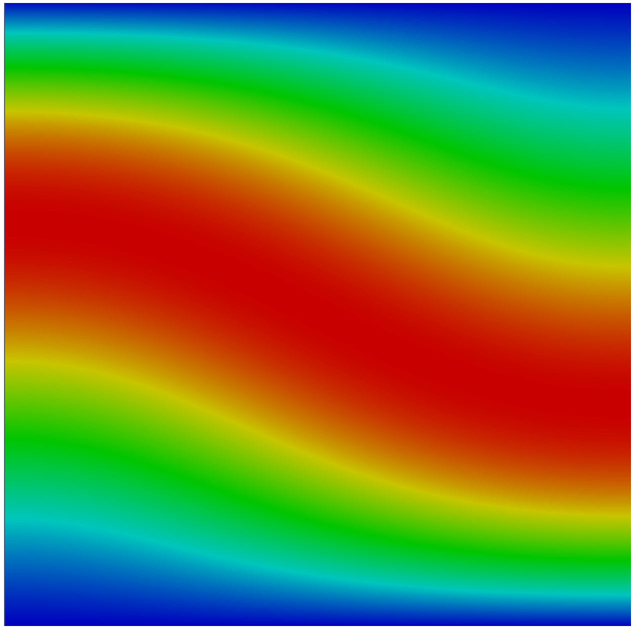
$$\mathcal{L} := \{q \in L^2(\Omega)\ /\ \nabla_{||} q \in L^2(\Omega) \text{ and } q|_{\Gamma_{in}} = 0\}.$$

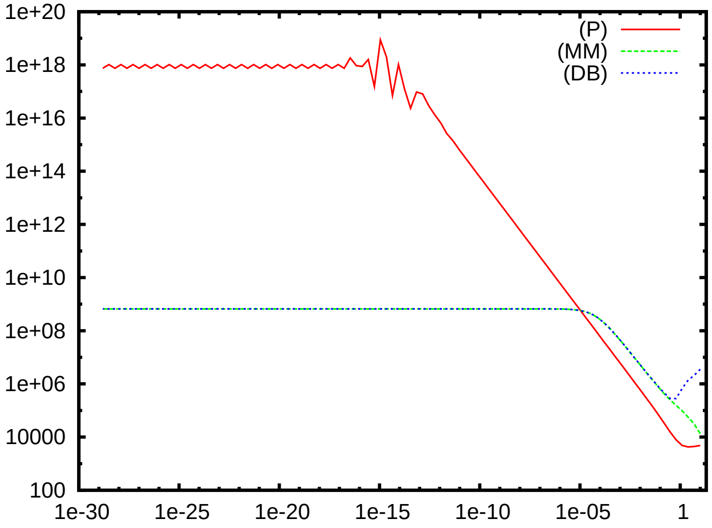
$$(AP_\varepsilon) \begin{cases} \int_\Omega A_\perp \nabla_\perp u^\varepsilon \cdot \nabla_\perp v\, dx + \int_\Omega A_{||} \nabla_{||} q^\varepsilon \cdot \nabla_{||} v\, dx = \int_\Omega f v\, dx, & \forall v \in \mathcal{V} \\ \int_\Omega A_{||} \nabla_{||} u^\varepsilon \cdot \nabla_{||} w\, dx - \int_\Omega \varepsilon A_{||} \nabla_{||} q^\varepsilon \cdot \nabla_{||} w\, dx = 0, & \forall w \in \mathcal{L} \end{cases}$$

- Reformulation of (P_ε) in a saddle-point problem (AP_ε)
- q^0 is (in the limit) a sort of Lagrange multiplier for the constraint $\nabla_{||} u^0 = 0$
- AP-formulation converges uniformely in ε towards the Limit-model (L)

- Exact solution u_e^ε, Num. sol. (AP) u_A^ε, Num. sol. (P) u_P^ε, Num. sol. (L) u_L^ε

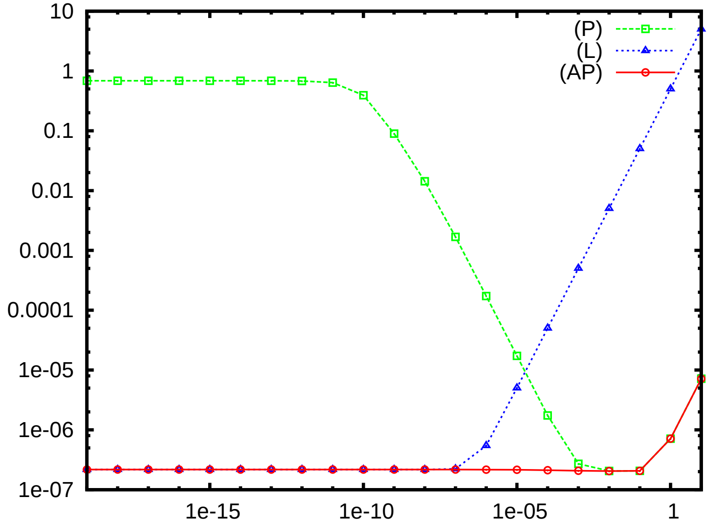
$$u_e^\varepsilon = \sin\left(\pi y + \alpha(y^2 - y)\cos(\pi x)\right) + \varepsilon \cos(2\pi x)\sin(\pi y)$$

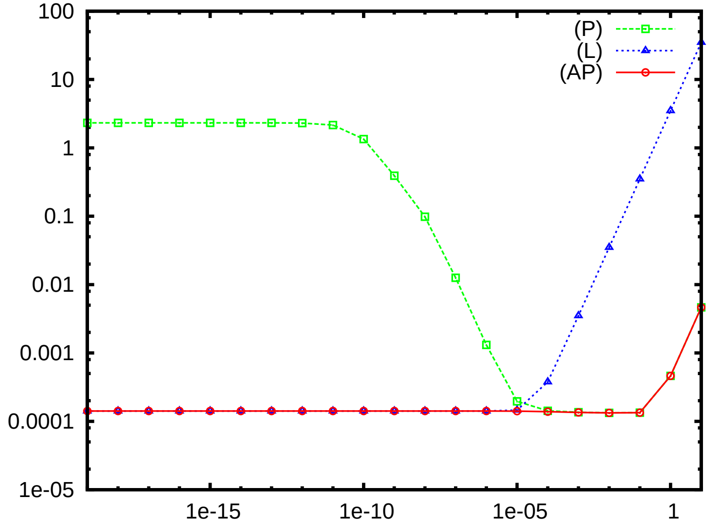




$\rightarrow$ Cond. of the discretization matrix of pb. (P) degenerates if $\varepsilon \rightarrow 0$, whereas it remains ε-independent for the AP-scheme

Numerical results





- The AP-scheme is unif. precise in ε, of order 3 in the L^2-norm and of order 2 in the H^1-norm ($\mathbb{Q}_2$-FE)
- This AP-scheme does not require to adapt the grid with respect to the field b
- The AP-formulation can treat variable anisotropies

Anisotropic parabolic equation

Work based on:

[1] A. Mentrelli, C. Negulescu "Asymptotic-Preserving scheme for highly anisotropic non-linear diffusion equations", Journal of Comp. Phys.
[2] A. Lozinski, J. Narski, C. Negulescu "Highly anisotropic temperature balance equation and its asymptotic-preserving resolution", submitted to M2AN.

- Two-fluid description of plasma dynamics

$$\begin{cases} \partial_t n_\alpha + \nabla \cdot (n_\alpha u_\alpha) = S_{n\alpha}\,, \\ m_\alpha n_\alpha \left[\partial_t u_\alpha + (u_\alpha \cdot \nabla) u_\alpha\right] = n_\alpha e_\alpha (E + u_\alpha \times B) - \nabla \cdot P_\alpha + R_\alpha\,, \\ \frac{3}{2} n_\alpha k_B \left[\partial_t T_\alpha + (u_\alpha \cdot \nabla) T_\alpha\right] = -\nabla \cdot q_\alpha - P_\alpha : \nabla u_\alpha + Q_\alpha\,, \end{cases}$$

- Fourier law: $q_\alpha := -\kappa_\alpha \nabla T_\alpha$
- Anisotropy due to the magn. field: $\kappa_{\alpha,||} \sim T_\alpha^{5/2}$, $\kappa_{\alpha,\perp}$ indep. on T_α

Theme: Efficient numerical resolution of temperature equation

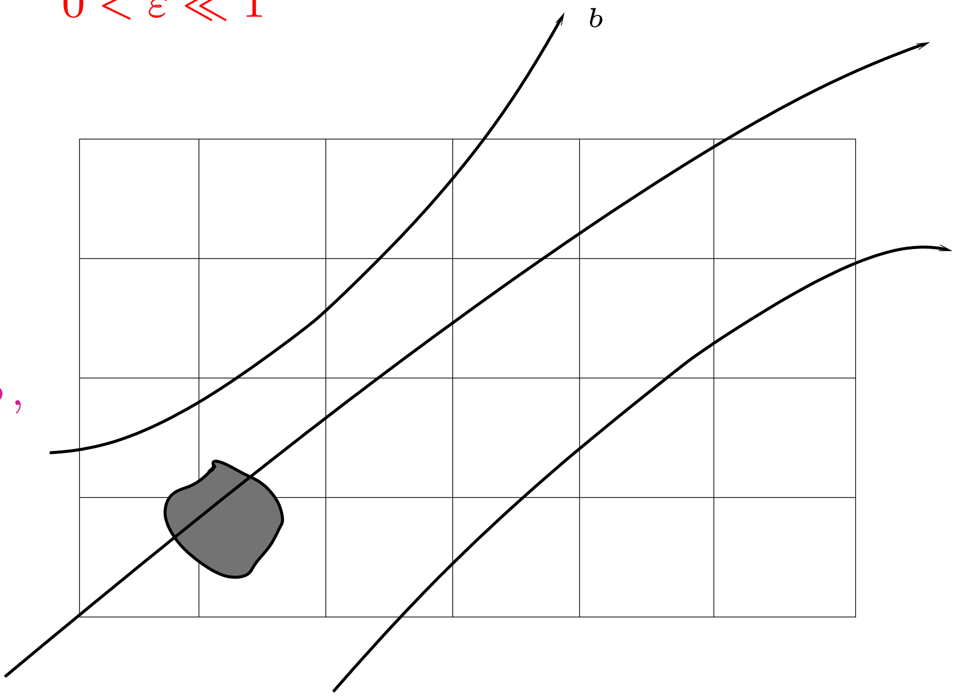
$$\partial_t \tilde{T} - \frac{1}{\varepsilon} \nabla_{||} \cdot (K_{||} \tilde{T}^{5/2} \nabla_{||} \tilde{T}) - \nabla_\perp \cdot (K_\perp \nabla_\perp \tilde{T}) = 0,$$

where $\tilde{T} := \frac{T}{||T||_\infty}$ and $\varepsilon := \frac{1}{||T||_\infty^{5/2}} \ll 1$ $\rightarrow$ Anisotropic, degenerate nonlinear parabolic equation

The 2D temperature balance equation

$$
(P_\varepsilon)\quad \begin{cases} \partial_t T - \frac{1}{\varepsilon}\nabla_{||}\cdot(K_{||}T^{5/2}\nabla_{||}T) - \nabla_\perp\cdot(K_\perp\nabla_\perp T) = 0\,, & \text{in}\quad [0,S]\times\Omega\,, \\ \frac{1}{\varepsilon} n_{||}\cdot(K_{||}T^{5/2}(t,\cdot)\nabla_{||}T(t,\cdot)) + n_\perp\cdot(K_\perp\nabla_\perp T(t,\cdot)) = -\gamma T(t,\cdot)\,, & \text{on}\quad [0,S]\times\Gamma_\perp\,, \\ \nabla_\perp T(t,\cdot) = 0\,, & \text{on}\quad [0,S]\times\Gamma_{||}\,, \\ T(0,\cdot) = T_0(\cdot)\,, & \text{in}\quad \Omega\,. \end{cases}
$$

$0 < \varepsilon \ll 1$



$$v_{||} := (v\cdot b)b\,, \qquad v_\perp := (Id - b\otimes b)v\,,$$

$$\nabla_{||}\phi := (b\cdot\nabla\phi)b\,, \qquad \nabla_\perp\phi := (Id - b\otimes b)\nabla\phi\,,$$

$$\nabla_{||}\cdot v := \nabla\cdot v_{||}\,, \qquad \nabla_\perp\cdot v := \nabla\cdot v_\perp\,.$$

$$\Gamma_{||} := \{x\in\Gamma \;/\; b(x)\cdot n(x) = 0\}\,,$$

$$\Gamma_\perp = \Gamma_{in}\cup\Gamma_{out} := \{x\in\Gamma \;/\; b(x)\cdot n(x) < 0\}\cup\{x\in\Gamma \;/\; b(x)\cdot n(x) > 0\}\,.$$

Numerical difficulties

- Putting formally $\varepsilon = 0$ in (P_ε), yields

$$(R) \quad \begin{cases} -\nabla_{||} \cdot (K_{||} T^{5/2} \nabla_{||} T) = 0\,, & \text{in} \quad [0,S] \times \Omega\,, \\ n_{||} \cdot (K_{||} T^{5/2}(t,\cdot) \nabla_{||} T(t,\cdot)) = 0\,, & \text{on} \quad [0,S] \times \Gamma_\perp\,, \\ \nabla_\perp T(t,\cdot) = 0\,, & \text{on} \quad [0,S] \times \Gamma_{||}\,, \\ T(0,\cdot) = T^0(\cdot)\,, & \text{in} \quad \Omega\,. \end{cases}$$

$\rightarrow$ (R) is an ill-posed pb., admitting infinitly many solutions!

$\rightarrow$ (P_ε) is a so-called singularly perturbed problem

Aim: Development of an asymp.-preserv. scheme for the resol. of (P_ε), which is

- accurate independent on ε
- capable to capture the limit model (P_0), for $\varepsilon \rightarrow 0$
- functional on cartesian grids, which have not to be adapted to the field lines

Mathematical results (Weak solution)

- More general formulation ($A_{||}$, $A_\perp$ satisfy pos., boundedness + coercivity cond.)

$$(P_m)\quad \begin{cases} \partial_t u - \nabla_{||}\cdot(A_{||}|u|^{m-1}\nabla_{||}u) - \nabla_\perp\cdot(A_\perp\nabla_\perp u) = 0\,, & \text{in} \quad [0,S]\times\Omega\,, \\ A_{||}|u|^{m-1}n_{||}\cdot\nabla_{||}u + A_\perp n_\perp\cdot\nabla_\perp u = -\gamma\, u\,, & \text{on} \quad [0,S]\times\Gamma_\perp\,, \\ \nabla_\perp u = 0\,, & \text{on} \quad [0,S]\times\Gamma_{||}\,, \\ u(0,\cdot) = u^0(\cdot)\,, & \text{in} \quad \Omega\,, \end{cases}$$

- Weak solution: Let $u^0 \in L^\infty(\Omega)$, $Q_T := (0,T)\times\Omega$, $\mathcal{V} := H^1(\Omega)$, $\mathcal{D} = L^2(0,T;\mathcal{V})$

$$\mathcal{W} := \{u \in L^\infty(Q_\infty),\ \text{such that } \forall\, T>0$$
$$\nabla_\perp u \in L^2(Q_T)\,, \quad |u|^{m-1}\nabla_{||}u \in L^2(Q_T)\,, \quad \partial_t u \in L^2(0,T;\mathcal{V}^*)\}\,.$$

$u \in \mathcal{W}$ is called a weak solution of (P_m), if $u(0,\cdot) = u^0$ and if $\forall\, T > 0$:

$$\int_0^T \langle \partial_t u(t,\cdot), \phi(t,\cdot)\rangle_{\mathcal{V}^*,\mathcal{V}}\, dt + \int_0^T\int_\Omega A_{||}|u|^{m-1}\nabla_{||}u\cdot\nabla_{||}\phi\, dxdt$$
$$+ \int_0^T\int_\Omega A_\perp\nabla_\perp u\cdot\nabla_\perp\phi\, dxdt + \gamma\int_0^T\int_{\Gamma_\perp} u\phi\, d\sigma\, dt = 0, \quad \forall\phi\in\mathcal{D}$$

Mathematical results (Ex./Unique/Pos.)

- Theorem: Let $m \geq 1$, $u^0 \in L^\infty(\Omega)$ and $0 < \beta \leq u^0 \leq M < \infty$ on Ω

$\Rightarrow \exists!$ weak solution $u \in \mathcal{W}$ of (P_m), satisfying $ce^{-Kt} \leq u \leq M$ a.e. on Q_∞, with a suff. small $c > 0$ and a suff. large $K > 0$.

- Proof:
  - Regularization + fixed point argument:
    $a_\alpha(u) := [\alpha + \min(|u|, M)]^{m-1}$ for fixed $0 < \alpha < 1$
    $\Rightarrow \exists! u_\alpha \in W_2^1(0, S; H^1(\Omega), L^2(\Omega))$
  - A priori estimates: indep. on α
  - Passage to the limit: $\alpha \to 0$ $\Rightarrow$ existence of $u \in \mathcal{W}$
  - Positivity and uniqueness (Comparision principle + Construction of a weak sub-solution)

Semi-discretization in space

- Singularly perturbed problem: Find $T(t,\cdot) \in \mathcal{V} := H^1(\Omega)$

$$(P_\varepsilon) \begin{cases} \langle \partial_t T(t,\cdot),\, v \rangle_{\mathcal{V}^*,\mathcal{V}} + \frac{1}{\varepsilon} \int_\Omega K_{||} |T|^{5/2} \nabla_{||} T(t,\cdot) \cdot \nabla_{||} v \, dx \\ \qquad + \int_\Omega K_\perp \nabla_\perp T(t,\cdot) \cdot \nabla_\perp v \, dx + \gamma \int_{\Gamma_\perp} T(t,\cdot) v \, d\sigma = 0, \quad \forall v \in \mathcal{V} \end{cases}$$

- Asymp.-Preserv. reform.: Find $(T(t,\cdot), q(t,\cdot)) \in \mathcal{V} \times \mathcal{L}$

$$(AP) \begin{cases} \langle \partial_t T, v \rangle_{\mathcal{V}^*,\mathcal{V}} + \int_\Omega (K_\perp \nabla_\perp T) \cdot \nabla_\perp v \, dx + \int_\Omega K_{||} \nabla_{||} q \cdot \nabla_{||} v \, dx + \gamma \int_{\Gamma_N} T v \, d\sigma = 0\,, \\ \qquad \forall v \in \mathcal{V} \\ \int_\Omega K_{||} T^{5/2} \nabla_{||} T \cdot \nabla_{||} w \, dx - \int_\Omega \varepsilon K_{||} \nabla_{||} q \cdot \nabla_{||} w \, dx = 0, \quad \forall w \in \mathcal{L}\,. \end{cases}$$

Idea: Introduction of auxiliary variable $q_\varepsilon \in \mathcal{L}$, such that $\nabla_{||} q_\varepsilon = \frac{1}{\varepsilon} T_\varepsilon^{5/2} \nabla_{||} T_\varepsilon$

$$\mathcal{L} := \{ q \in L^2(\Omega) \ / \ \nabla_{||} q \in L^2(\Omega) \text{ and } q|_{\Gamma_{in}} = 0 \}.$$

- Putting formally $\varepsilon = 0$ in (AP), yields

$$(L) \quad \begin{cases} \langle \partial_t T,\, v\rangle_{\mathcal{V}^*,\mathcal{V}} + \int_\Omega (K_\perp \nabla_\perp T)\cdot \nabla_\perp v\, dx + \int_\Omega K_{\|} \nabla_{\|} q \cdot \nabla_{\|} v\, dx + \gamma \int_{\Gamma_\perp} T v\, ds = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall v \in \mathcal{V} \\ \int_\Omega K_{\|} T^{5/2} \nabla_{\|} T \cdot \nabla_{\|} w\, dx = 0, \quad \forall w \in \mathcal{L} \end{cases}$$

- Limit-pb. is a well-posed saddle point problem

➠ q acts as a Lagrangian for the constraint $T(t,\cdot) \in \mathcal{G}$

$$\mathcal{G} := \{p \in \mathcal{V} \;/\; \nabla_{\|} p = 0 \text{ in } \Omega\}$$

➠ this q provides the uniqueness of the solution

Indeed, the sequence $T^\varepsilon(t,\cdot)$ tends in the limit $\varepsilon \to 0$ towards the sol. of

$$(L) \;\; \langle \partial_t T(t,\cdot),\, v\rangle_{\mathcal{V}^*,\mathcal{V}} + \int_\Omega K_\perp \nabla_\perp T(t,\cdot) \cdot \nabla_\perp v\, dx + \gamma \int_{\Gamma_\perp} T(t,\cdot) v\, d\sigma = 0, \quad \forall v \in \mathcal{G}$$

Semi-discretization in time (Euler implicit)

$$(AP)\quad \begin{cases} \langle \partial_t T, v\rangle_{\mathcal{V}^*,\mathcal{V}} + \int_\Omega (K_\perp \nabla_\perp T)\cdot \nabla_\perp v\, dx + \int_\Omega K_{||}\nabla_{||} q \cdot \nabla_{||} v\, dx + \gamma \int_{\Gamma_N} T v\, d\sigma = 0\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall v \in \mathcal{V} \\ \int_\Omega K_{||} T^{5/2} \nabla_{||} T \cdot \nabla_{||} w\, dx - \int_\Omega \varepsilon K_{||} \nabla_{||} q \cdot \nabla_{||} w\, dx = 0, \quad \forall w \in \mathcal{L}\,. \end{cases}$$

$$(\Theta,\chi) := \int_\Omega \Theta \chi\, dx\,, \quad a_{\|nl}(\Psi,\Theta,\chi) := \int_\Omega K_\| \Psi^{5/2} \nabla_\| \Theta \cdot \nabla_\| \chi\, dx\,,$$

$$a_\|(\Theta,\chi) := \int_\Omega K_\| \nabla_\| \Theta \cdot \nabla_\| \chi\, dx\,, \qquad a_\perp(\Theta,\chi) := \int_\Omega K_\perp \nabla_\perp \Theta \cdot \nabla_\perp \chi\, dx\,,$$

Find $(T_h^{n+1}, q_h^{n+1}) \in \mathcal{V}_h \times \mathcal{L}_h \subset \mathcal{V} \times \mathcal{L}$, solution of:

$$(E_{AP})\quad \begin{cases} (T_h^{n+1}, v_h) + \tau \left(a_\perp(T_h^{n+1}, v_h) + a_\|(q_h^{n+1}, v_h) + \gamma \int_{\Gamma_\perp} T_h^{n+1} v_h\, ds \right) = (T_h^n, v_h) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall v_h \in \mathcal{V}_h \\ a_{\|nl}(T_h^n, T_h^{n+1}, w_h) - \varepsilon a_\|(q_h^{n+1}, w_h) = 0\,, \quad \forall w_h \in \mathcal{L}_h\,. \end{cases}$$

Semi-discretization in time

- Implicit Euler time-discretization:
  - ➠ first order scheme in time + AP-scheme
- Crank-Nicolson time-discretization:
  - ➠ second order scheme in time
  - ➠ A-stable, but not L-stable $\Rightarrow$ not AP !
  - ➠ restrictive time-step $\Delta t \sim \frac{\varepsilon}{(T^n)^{5/2}}$
- Diagonally implicit Runge-Kutta (DIRK) time-discretization:
  - ➠ second order scheme in time
  - ➠ A-stable and L-stable
  - ➠ 2 syst. to be solved $\Rightarrow$ 2 times slower than CN, but AP !

Numerical results

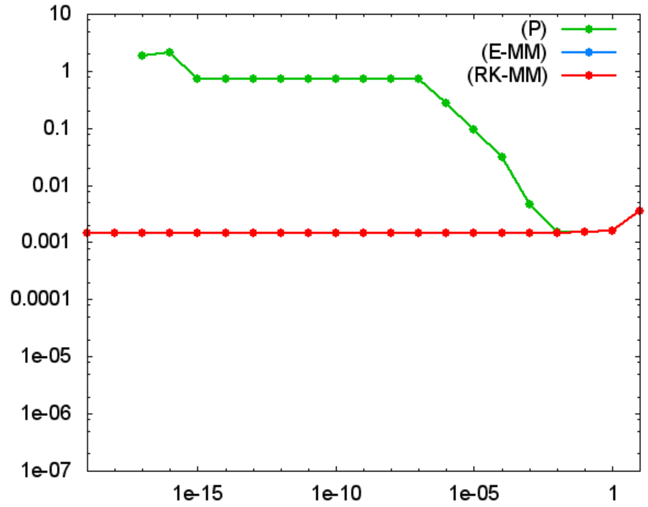
- Magnetic field: $b = \frac{B}{|B|}$, $B = \begin{pmatrix} \alpha(2y-1)\cos(\pi x) + \pi \\ \pi\alpha(y^2 - y)\sin(\pi x) \end{pmatrix}$
- Initial condition: (a) Constr. of analytic solution

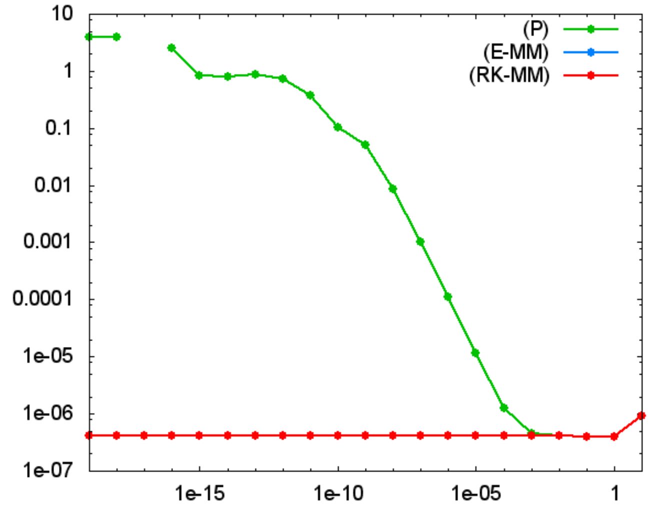
  (b) Gaussian peak:

$$T(t=0,x,y) = \frac{T_m}{2}\left(1 + e^{-50(x-0.5)^2 - 50(y-0.5)^2}\right),$$

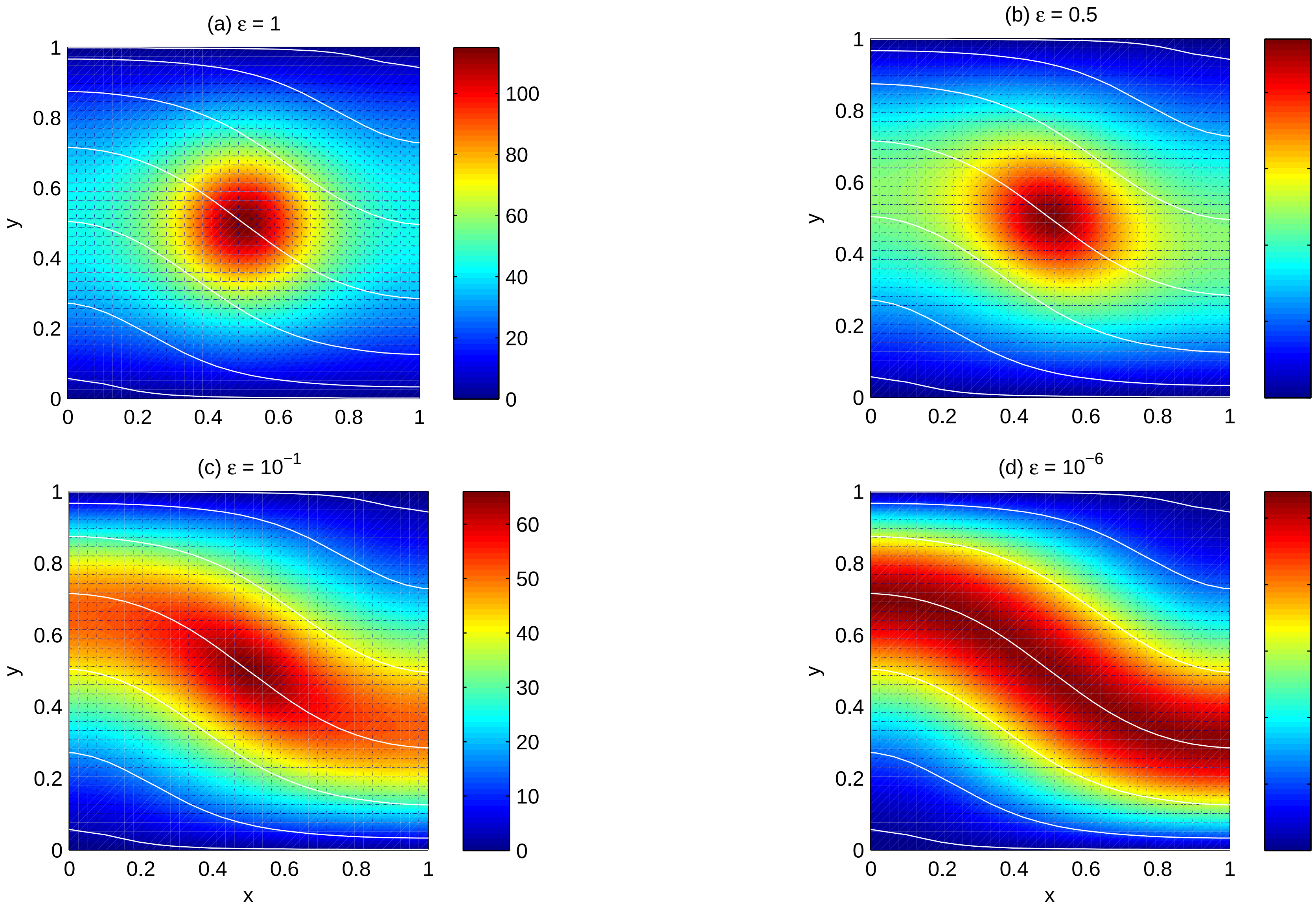
- Cartesian grids, finite element method ($\mathbb{Q}_2$-FEM)

L^2-errors between the exact and num. sol. as a function of ε





Numerical results



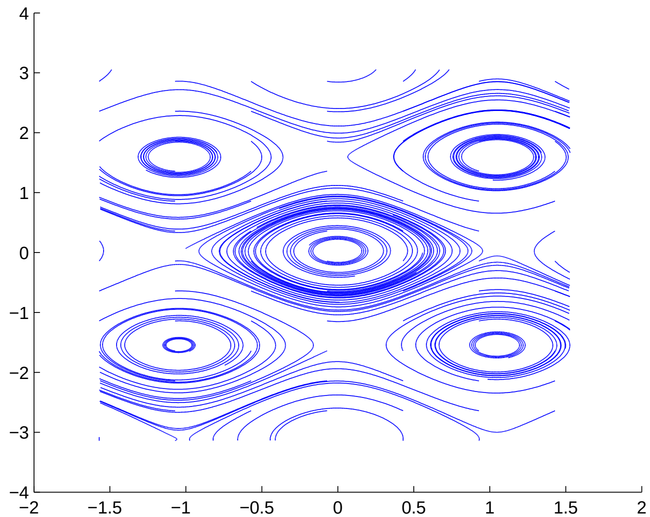
Closed magnetic field lines

• Magn. field lines: can be closed in real tokamak plasma simulations
$\Rightarrow$ Difficulties in using previous AP-scheme, due to determination of auxiliary var. q_ε such that $\nabla_{||} q_\varepsilon = \frac{1}{\varepsilon} T_\varepsilon^{5/2} \nabla_{||} T_\varepsilon$

$$\mathcal{L} := \{q \in L^2(\Omega) \ / \ \nabla_{||} q \in L^2(\Omega) \text{ and } q|_{\Gamma_{in}} = 0\}.$$

• Example of magn. field lines:

$$B = \nabla \times (\psi(x,y) e_z) + B(x,y) e_z \,, \quad \psi(x,y) = cos(x) + A cos(y - \omega t)$$



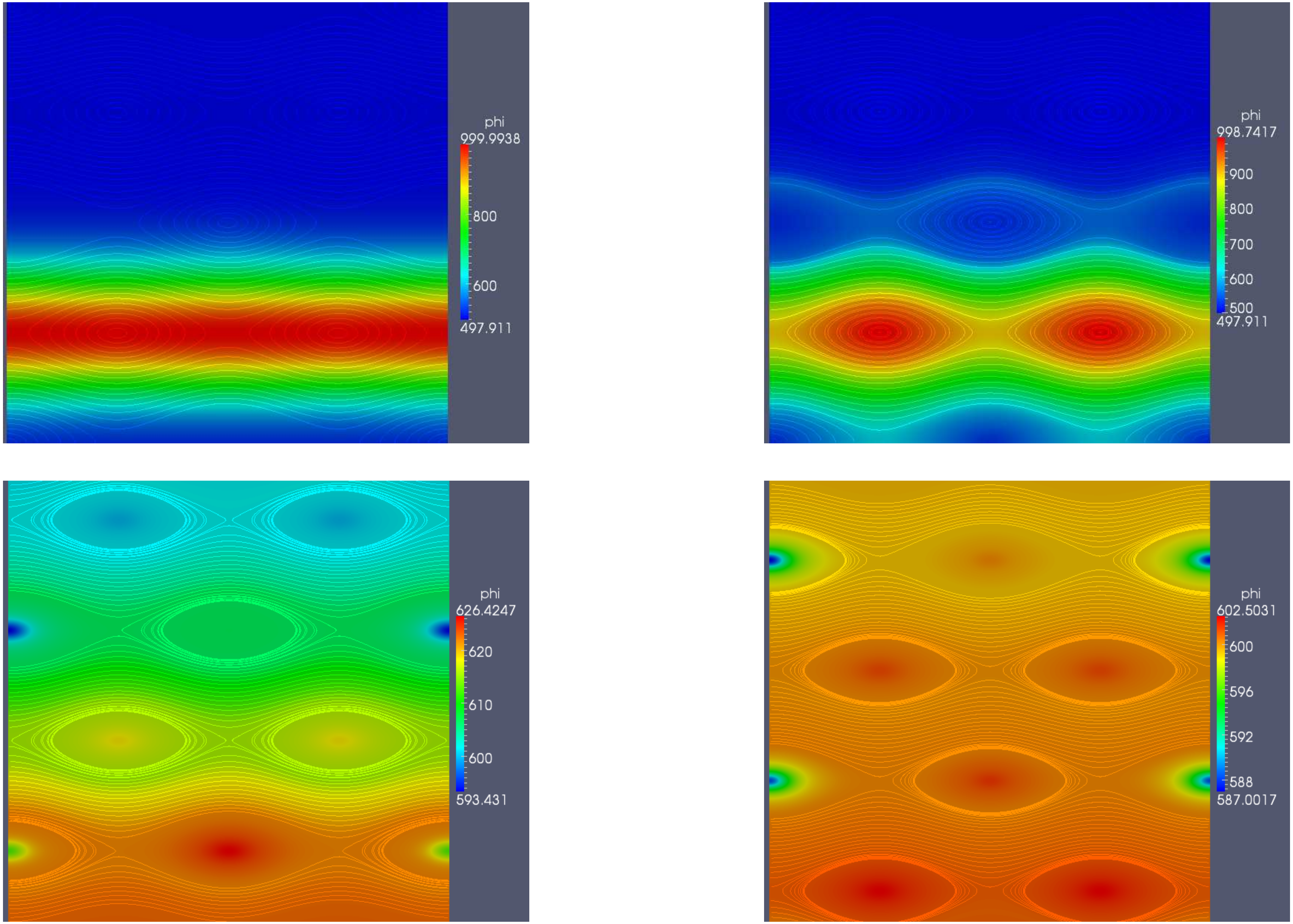
- Idea: Introduction of a stabilization term

$$(AP)_h^1 \quad \begin{cases} \langle \partial_t T_h, v_h \rangle + \int_\Omega (K_\perp \nabla_\perp T_h) \cdot \nabla_\perp v_h \, dx + \int_\Omega K_\parallel q_h \cdot \nabla_\parallel v_h \, dx + \gamma \int_{\Gamma_\perp} T_h v_h \, ds = 0, \\ \qquad\qquad \forall v_h \in \mathcal{V}_h \\ \int_\Omega K_\parallel T_h^{5/2} \nabla_\parallel T_h \cdot w_h \, dx - \varepsilon \int_\Omega K_\parallel q_h w_h \, dx = 0 \end{cases}$$

$$(AP)_h^2 \quad \begin{cases} \langle \partial_t T_h, v_h \rangle + \int_\Omega (K_\perp \nabla_\perp T_h) \cdot \nabla_\perp v_h \, dx + \int_\Omega K_\parallel \nabla_\parallel q_h \cdot \nabla_\parallel v_h \, dx + \gamma \int_{\Gamma_\perp} T_h v_h \, ds = 0, \\ \qquad\qquad \forall v_h \in \mathcal{V}_h \\ \int_\Omega K_\parallel T_h^{5/2} \nabla_\parallel T_h \cdot \nabla_\parallel w_h \, dx - \varepsilon \int_\Omega K_\parallel \nabla_\parallel q_h \cdot \nabla_\parallel w_h \, dx = h^3 \int_\Omega q_h w_h \, dx, \end{cases}$$

- Advantages:
  - ➠ permits to determine uniquely q_h, without imposing Dirichlet B.C. on the inflow boundary Γ_{in}
  - ➠ permits to treat closed field lines

Numerical results



Conclusions

- Singularly perturbed problems:
  - contain small parameters, that lead to various asymptotic regimes
  - classical schemes become too expensive, and even "unusable" in the limit regime
- Asymptotic-Preserving methodology:
  - offers simple, robust and efficient num. meth. for large class of singularly perturbed pb.
  - preserves at discrete level the limit asymptotics
  - solves the microscale, and automatically switches to a macroscopic solver for the limit pb.