Generalized model solutions for the amplification of electromagnetic waves in overdense plasma physics

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Institute fur Plasma Physics, Garching, October 15th, 2013,
ESF Workshop

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Outline

Original electromagnetic equations

Turning point theory for 2-2 systems

Conclusion

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Equations for the electrons and the electromagnetic field

Equations shown in the paper of White and Chen.
System of equations, $B_0 = |B_0| e_z$:

$$\begin{cases} \nabla \wedge E = i\omega B \\ c^2 \nabla \wedge B = -4\pi n_0(x) e v - i\omega E \\ -i\omega J = \epsilon_0 (\omega_p(x))^2 E - \omega_c J \wedge e_z - \nu J \end{cases}$$

where $J = -4\pi\epsilon_0 n_0(x) e v$:

$$(\omega_p(x))^2 = \frac{4\pi e^2}{m} n_0(x)$$

is the plasma oscillation frequency of electrons,

$$\omega_c = \frac{e|B_0|}{m}$$

is the cyclotron frequency.
Note the classical equation $c^2 \nabla \wedge B = \frac{1}{\epsilon_0} J - i\omega E$.

System of Chen and White (II)

Solutions with $e^{ik_0 \sin\theta_0 y}$. (Properties of invariance of p_y because no dependency of the symbol in y). Total system

$$\begin{cases} i\omega B_1 = ik_0 \sin\theta_0 E_3 \\ i\omega B_2 = -E_3' \\ i\omega B_3 = E_2' - ik_0 \sin\theta_0 E_1 \\ c^2(ik_0 \sin\theta_0 B_3) = j_1 - i\omega E_1 \\ c^2(-B_3') = j_2 - i\omega E_2 \\ c^2(B_2' - ik_0 \sin\theta_0 B_1) = j_3 - i\omega E_3 \\ -i\omega j_1 = \omega_p^2(x)E_1 - \omega_c j_2 - \nu j_1 \\ -i\omega j_2 = \omega_p^2(x)E_2 + \omega_c j_1 - \nu j_2 \\ -i\omega j_3 = \omega_p^2(x)E_3 - \nu j_3 \end{cases}$$

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\begin{cases}
i\omega B_2 = -E_3' \\
c^2(B_2' - ik_0 \sin\theta_0 B_1) = j_3 - i\omega E_3 \\
\\
i\omega B_3 = E_2' - ik_0 \sin\theta_0 E_1 \\
c^2(-B_3') = j_2 - i\omega E_2 \\
\\
c^2(ik_0 \sin\theta_0 B_3) = j_1 - i\omega E_1 \\
-i\omega j_1 = \omega_p^2(x) E_1 - \omega_c j_2 - \nu j_1 \\
-i\omega j_2 = \omega_p^2(x) E_2 + \omega_c j_1 - \nu j_2 \\
\\
-i\omega j_3 = \omega_p^2(x) E_3 - \nu j_3 \\
i\omega B_1 = ik_0 \sin\theta_0 E_3
\end{cases}
$$

System of Chen and White (III)

Solutions with $e^{ik_0 \sin\theta_0 y}$. (Properties of invariance of p_y because no dependency of the symbol in y). Total system

$$
\begin{cases}
i\omega B_2 = -E_3' \\
c^2(B_2' - ik_0 \sin\theta_0 B_1) = j_3 - i\omega E_3 \\
\\
i\omega B_3 = E_2' - ik_0 \sin\theta_0 E_1 \\
c^2(-B_3') = j_2 - i\omega E_2 \\
\\
j_1 - i\omega E_1 = c^2(ik_0 \sin\theta_0 B_3) \\
\omega_p^2(x) E_1 - \omega_c j_2 + (i\omega - \nu) j_1 = 0 \\
-\omega_c j_1 - (i\omega - \nu) j_2 = \omega_p^2(x) E_2 \\
\\
-i\omega j_3 = \omega_p^2(x) E_3 - \nu j_3 \\
i\omega B_1 = ik_0 \sin\theta_0 E_3
\end{cases}
$$

Replacing j_3 and $cB_1 (= \sin\theta_0 E_3)$ in terms of E_3, one obtains

$$\begin{cases} E_3' = -ik_0(cB_2) \\ (cB_2)' = ik_0[\sin^2\theta_0 - 1 + \frac{(\omega_p(x))^2}{i\omega(-i\omega+\nu)}]E_3 \end{cases}$$

Ordinary modes. Current density $j_3 = (-i\omega + \nu)^{-1}(\omega_p(x))^2 E_3$.

The rest of the system contains E_2', $(cB_3)'$. One seeks (j_1, j_2, E_1) in terms of E_2, cB_3. The system is

$$\begin{cases} (i\omega - \nu)j_1 & -\omega_c j_2 & +\omega_p^2(x)E_1 & = 0 \\ \omega_c j_1 & +(i\omega-\nu)j_2 & & = -\omega_p^2(x)E_2 \\ \frac{1}{i\omega}j_1 & & -E_1 & = \sin\theta_0 cB_3 \end{cases}$$

Determinant $d_\nu = (\omega_h^2(x) + \nu^2 - \omega^2) + \frac{\nu}{i\omega}(2\omega^2 - \omega_p^2(x))$, with $\omega_h^2(x) = \omega_c^2 + \omega_p^2(x)$.

Apart from the root $\omega = 0$, root $\omega - \omega_h \simeq i\nu(\frac{\omega_p^2}{2\omega_h^2} - 1)$.

On (j_1, j_2):

$$\begin{cases} (i\omega - \nu + \frac{\omega_p^2}{i\omega})j_1 - \omega_c j_2 = \sin\theta_0\omega_p^2(cB_3) \\ \omega_c j_1 + (i\omega - \nu)j_2 = -\omega_p^2 E_2 \end{cases}$$

with $E_1 = -\sin\theta_0 cB_3 + (i\omega)^{-1}j_1$.

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Resulting system from

$$\begin{cases} E_2' = ik_0[\frac{\sin\theta_0}{i\omega} j_1 + (1-\sin^2\theta_0)cB_3] \\ (cB_3)' = ik_0[E_2 - (i\omega)^{-1} j_2] \end{cases}$$

yields

$$E_2' = ik_0[-\frac{\omega_c\omega_p^2}{i\omega d_\nu}\sin\theta_0 E_2 + (1-\sin^2\theta_0(1-\frac{(i\omega-\nu)\omega_p^2}{i\omega d_\nu}))cB_3]$$

$$(cB_3)' = ik_0[(1+\frac{\omega_p^2(i\omega-\nu+\frac{\omega_p^2}{i\omega})}{i\omega d_\nu})E_2 + \frac{\sin\theta_0\omega_c\omega_p^2}{i\omega d_\nu}cB_3].$$

Recall the electric current j:

$$\begin{cases} d_\nu j_1 = \omega_p^2[-\omega_c E_2 + (i\omega-\nu)\sin\theta_0(cB_3)] \\ d_\nu j_2 = -\omega_p^2[(i\omega-\nu+\frac{\omega_p^2}{i\omega})E_2 + \omega_c\sin\theta_0(cB_3)] \end{cases}$$

No need for an extra differential equation for E_1, thanks to:

$$E_1 = -\sin\theta_0 cB_3 + (i\omega)^{-1} j_1.$$

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Structure

$$\begin{pmatrix} E_3 \\ cB_2 \end{pmatrix}' = ik_0 M_o(x) \begin{pmatrix} E_3 \\ cB_2 \end{pmatrix}, \begin{pmatrix} E_2 \\ cB_3 \end{pmatrix}' = ik_0 M_X(x) \begin{pmatrix} E_3 \\ cB_2 \end{pmatrix}.$$

where

$$M_o(x) = \begin{pmatrix} 0 & -1 \\ \sin^2\theta_0 - 1 + \frac{\omega_p^2}{\omega^2 + i\omega\nu} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ \sin^2\theta_0 - \epsilon_\nu(x) & 0 \end{pmatrix},$$

$$M_X = \begin{pmatrix} -a_\nu(x) & b_\nu(x) \\ c_\nu(x) & a_\nu(x) \end{pmatrix}.$$

where the coefficients of M_X have a simple pole at $\omega = \omega_h$ when $\nu = 0$. and detM_X has also a simple pole.

Turning points

Ordinary mode $E_3'' = k_0^2(1 - \sin^2\theta_0 - \frac{\omega_p^2(x)}{\omega^2 + i\omega\nu})E_3$
(no additional issue, classical turning point analysis).
Assume that $\omega_p(x)^2$ is strictly increasing and that there exists a unique point x_0 such that $\omega_p^2(x_0) = \omega^2\cos^2\theta_0$
Complex phase ρ_ν solution of

$$-(\rho_\nu'(x))^2\rho_\nu(x) = 1 - \sin^2\theta_0 - \frac{\omega_p^2(x)}{\omega^2 + i\omega\nu} = \epsilon(x) - \sin^2\theta_0 + \frac{i\omega_p}{\omega(\omega^2 + i\omega\nu)}\nu$$

Let x_ν be the unique point (ν small) such that

$$\epsilon_\nu(x_\nu) = \sin^2\theta_0$$

Then
$\rho_\nu(x) = r_\nu(x)(x - x_\nu)$, with $r_\nu(x_\nu) \neq 0$.

$$r_\nu(x) = \left(\int_0^2 s^{\frac{1}{2}} \frac{\partial_x\omega_p^2(x_\nu + s(x - x_\nu))}{\omega^2 + i\omega\nu} ds\right)^{\frac{2}{3}}$$

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Reduction of the system (discussions with M. Williams)

Equation:

$$E'' - 3 = -k_0^2(\rho_\nu'(x))^2\rho_\nu(x)E_3$$

New variable $X = \rho_\nu(x)$

$$X'(x)\frac{d}{dX}(X'(x)\frac{dE_3}{dX}) = -k_0^2(X'(x))^2X(x)E_3$$

Introduce $w = (X'(x))^{\frac{1}{2}}E_3$,

$$(X'(x))^{\frac{1}{2}}\frac{d^2}{dX^2}((X'(x))^{\frac{1}{2}}E_3) = \frac{d}{dX}(X'(x)\frac{dE_3}{dX}) + (X'(x))^{\frac{1}{2}}\frac{d^2}{dX^2}((X'(x))^{\frac{1}{2}}$$

Rewrite then

$$\frac{d^2w}{dX^2} = -k_0^2(X + k_0^{-2}\theta(X))w \Leftrightarrow \frac{d^2w}{dT^2} = -(T + k_0^{-\frac{2}{3}}h(T))w$$

Wasov's conjugation lemma for k_0^{-2} small reduces exactly to the Airy equation.

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Plan

Extraordinary mode, normal incidence

Simplification $\sin\theta_0 = 0$

System $M_X^{\theta_0=0} = \begin{pmatrix} 0 & 1 \\ 1 + \frac{\omega_p^2(i\omega - \nu + \frac{\omega_p^2}{i\omega})}{i\omega d_\nu} & 0 \end{pmatrix}$.

Usual ODE:

$$E_2'' = -k_0^2 \epsilon_X(x) E_2,$$

Observe

$$\epsilon_X(x) = -\frac{(\omega_p^2(x) - \omega^2 - i\omega\nu)^2 - \omega^2\omega_c^2}{d_\nu \omega^2}$$

No additional issue for points such that $\epsilon_X(x) = 0$ for $\nu = 0$: it is still a turning point.

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No additional issue for points such that $\epsilon_X(x) = 0$ for $\nu = 0$: it is still a turning point.

Points where $\epsilon_X(x)$ is singular: $\epsilon_X(x) = \frac{k(x)}{x - x_X}$

$$\frac{d}{dx}\left(\frac{dE_2}{dx}\right) = -k_0^2 \frac{k(x_X)}{x - x_X}(1 + (x - x_X)^2 g((x - x_X)^2))E_2$$

New variable $x - x_X = y^2$ (right)

$$y^2 \frac{1}{2y}\frac{d}{dy}\left(\frac{1}{2y}\frac{dE_2}{dy}\right) = -k_0^2 k(x_X)(1 + y^2 g(y^2))E_2$$

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$$\tilde{E}_2 = C_1 Y_0(2k_0(k(x_X))^{\frac{1}{2}}(x - x_X)^{\frac{1}{2}}) + C_2 J_0(2k_0(k(x_X))^{\frac{1}{2}}(x - x_X)^{\frac{1}{2}}).$$

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$$U = \frac{1}{2\pi} \int \begin{pmatrix} \sigma_0(x,\theta) \\ \sigma_1(x,\theta) \end{pmatrix} e^{ik_0(\rho(x)\theta - \frac{\theta^3}{3})} d\theta.$$

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$$bc + a^2 = (\rho')^2\rho$$

Note that $\det(M_X^{\theta_0=0} - \rho'\theta Id) = (\rho'\theta)^2 - a^2 - bc.$

Determinant vanishes for $\theta^2 = \rho$. Eigenvalues $\lambda = -\rho'\theta \pm \rho'\rho^{\frac{1}{2}}$ (complex representation).

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Some partial conclusions

- A simplified version of the full system
- Bessel functions for the representation of extraordinary modes.

$\Rightarrow$ we recover the behavior in $\frac{1}{x-x_X}$ for a part of the solution (WIP).

- Airy function-type analysis for the turning point for the 2-2 system

$\Rightarrow$ Better than considering the second order ODE (s)? WIP.

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Original electromagnetic equations

Turning point theory for 2-2 systems

Conclusion

Appendix

Model studied in BLSS (I)

In the case of a scalar vector potential, we wrote

$$\nabla(\nabla A) + k_0^2(1 - N(x))A + i\nu k_0 A = 0.$$

The wave number k_0 was imposed by the scaling on the **adimensional** electronic density $N(x) \in [0, 1[$.
Equations supplemented by an incident known wave and a radiation boundary condition.
Classical WKB approximation (as presented also by O. Maj in October 2012):

$$A = (a_0 + (ik_0)^{-1}a_1 + ...)e^{ik_0\varphi}$$

Assumption $k_0 >> 1$.
Leading order term in k_0^2 vanishes:
$N(x) - 1 + |\nabla\varphi|^2 = 1$: **eikonal** equation (Hamilton-Jacobi).
Next order term in k_0^1 vanishes:
$\nu a_0 + 2\nabla a_0 \nabla\varphi + a_0 \nabla.(\nabla\varphi) = 0$: transport equation.
Rewrites as $\nu E + \text{div}(E\nabla\varphi) = 0$, $E = |a_0|^2$, νE is the absorbed laser energy.

Model BLSS (II)

We assume N depending only on x.
The model is supplemented by incident conditions on $\Gamma_{inc} = \{x = 0\}$:

$$a_0(0,y,z) = e^{ik_0 \sin\alpha(\cos\varphi y + \sin\varphi z)}$$
$$\nabla a_0(0,y,z) = (\cos\alpha, \sin\alpha\cos\varphi, \sin\alpha\sin\varphi) a_0(0,y,z)$$

Elementary example: $N(x) = x$:
Propagation of singularities:$\Rightarrow A(x,y,z) = A(x) e^{ik_0 \sin\alpha(\cos\varphi y + \sin\varphi z)}$,
Equation on A:

$$A'' + k_0^2(\cos^2\alpha - x)A + i\nu k_0 A = 0.$$

Writes

$$\begin{pmatrix} A \\ (ik_0)^{-1}A' \end{pmatrix}' = ik_0 \begin{pmatrix} 0 & 1 \\ \cos^2\alpha - x + \frac{i\nu}{k_0} & 0 \end{pmatrix} \begin{pmatrix} A \\ (ik_0)^{-1}A' \end{pmatrix}.$$

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$$\begin{pmatrix} A \\ (ik_0)^{-1}A' \end{pmatrix}' = ik_0 \begin{pmatrix} 0 & 1 \\ \cos^2\alpha - x + \frac{i\nu}{k_0} & 0 \end{pmatrix} \begin{pmatrix} A \\ (ik_0)^{-1}A' \end{pmatrix}.$$

Model of BLSS (III)

Explicit solutions

$$A = C_1 Ai(k_0^{\frac{2}{3}}(x - \cos^2\alpha - \frac{i\nu}{k_0})) + C_2 Bi(k_0^{\frac{2}{3}}(x - \cos^2\alpha - \frac{i\nu}{k_0}))$$

where Ai and Bi: pair of fundamental solutions of $u'' = zu$, Ai being the one which Fourier transform is $e^{i\frac{t^3}{3}}$, and Bi being another solution, $Bi \to +\infty$ at $+\infty$.
Behavior at $|z| \to \infty$, $|arg(x)| < \frac{2}{3}\pi$:

$$Ai(z) \simeq Kz^{-\frac{1}{4}} e^{-\frac{2}{3}z^{\frac{3}{2}}}, Bi(z) \simeq Kz^{-\frac{1}{4}} e^{\frac{2}{3}z^{\frac{3}{2}}}.$$

For $x - \cos^2\alpha > \sqrt{3}\frac{\nu}{k_0}$, one has $\Re\frac{2}{3}z^{\frac{3}{2}} \to +\infty$, radiation condition imply $C_2 = 0$.

$$A(x, y, z) = C_0 Ai(k_0^{\frac{2}{3}}(x - \cos^2\alpha - \frac{i\nu}{k_0}))e^{ik_0 \sin\alpha(\cos\varphi y + \sin\varphi z)}.$$

It is the 'ordinary mode' as described in Chen-White (eq (16), (17)).

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For $x = 0$,

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Model of BLSS (V): general function N

Interpretation: ray tracing $Y(s)$ s. t. $Y'(s) = \nabla\varphi(Y(s)) = P(s)$.

$$\frac{d}{ds}(a_0(Y(s)) = -a_0(Y(s))\Delta\varphi(Y(s)) - \nu a_0(Y(s))$$

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System of rays (bicharacteristics):
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Property (flavor of a theorem): The high frequency singularities of a solution of $Pu = 0$ belong to integral curves of the previous system.

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