Gaussian Beam Approximations of High Frequency Waves

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ESF Exploratory Workshop
IPP Garching, October 2013
High frequency waves

Cauchy problem for scalar wave equation

\[ u_{tt} - c(x)^2 \Delta u = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \]

\[ u(0, x) = A(x)e^{i\phi(x)/\varepsilon}, \quad u_t(0, x) = \frac{1}{\varepsilon} B(x)e^{i\phi(x)/\varepsilon}, \]

where \( c(x) \) (variable) smooth speed of propagation.
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where $c(x)$ (variable) smooth speed of propagation.

- **High frequency** →
  - short wave length →
  - highly oscillatory solutions →
  - many gridpoints.

High frequency waves

Direct numerical solution resolves wavelength:

$$\# \text{gridpoints} \sim \varepsilon^{-n} \text{ at least}$$

$$\Rightarrow \text{cost} \sim \varepsilon^{-n-1} \text{ at least}$$

Often unrealistic approach for applications in e.g. optics, electromagnetics, geophysics, acoustics, quantum mechanics, ...
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Cauchy problem for scalar wave equation

\[ u_{tt} - \nabla \cdot c^\varepsilon(x) \nabla u = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \]

\[ u(0, x) = A(x), \quad u_t(0, x) = B(x), \]

where \( c^\varepsilon(x) \in \mathbb{R}^{d\times d} \) has variations on length scale \( \sim \varepsilon \).

The functions \( A(x) \) and \( B(x) \) are smooth (and independent of \( \varepsilon \)).
High frequency material

Cauchy problem for scalar wave equation

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  Prohibitively expensive when \( \varepsilon \ll 1 \), particularly in higher dimensions.

Our approach: Heterogeneous Multiscale Method (HMM) [E, Engquist, 2001]. Solve small micro problems (localized in time and space) to probe effective dynamics, which is approximated on coarse grid \( (\Delta x \gg \varepsilon) \). Method cost (essentially) independent of \( \varepsilon \).
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Geometrical optics

Wave equation

\[ u_{tt} - c(x)^2 \Delta u = 0. \]

Write solution on the form

\[ u(t, x) = a(t, x, \varepsilon)e^{i\phi(t, x)/\varepsilon}. \]
Wave equation

$$u_{tt} - c(x)^2 \Delta u = 0.$$ 

Write solution on the form

$$u(t, x) = a(t, x, \varepsilon)e^{i\phi(t,x)}/\varepsilon.$$ 

(a) Amplitude $a(x)$

(b) Phase $\phi(x)$
Geometrical optics

- $a, \phi$ vary on a much coarser scale than $u$. (And varies little with $\varepsilon$.) Geometrical optics approximation considers $a$ and $\phi$ as $\varepsilon \to 0$. 

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0
\]
\[
\frac{\partial \phi}{\partial x} \frac{\partial a}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial a}{\partial y} + \frac{\partial \phi}{\partial x} \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial \phi}{\partial x} = 0
\]
a, φ vary on a much coarser scale than u. (And varies little with ε.) Geometrical optics approximation considers a and φ as ε → 0.

Phase and amplitude satisfy eikonal and transport equations

\[ \phi_t^2 - c(y)^2|\nabla \phi|^2 = 0, \quad a_t + c \frac{\nabla \phi \cdot \nabla a}{|\nabla \phi|} + \frac{c^2 \Delta \phi - \phi_{tt}}{|2c|\nabla \phi|} a = 0. \]
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\]

- Ray tracing: $x(t), p(t)$ bicharacteristics of the eikonal equation,

\[
\frac{dx}{dt} = c(x)^2 p, \quad \frac{dp}{dt} = -\frac{\nabla c(x)}{c(x)}, \quad \phi(t, x(t)) = \phi(0, x(0)).
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Geometrical optics

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\]

- Good accuracy for small $\varepsilon$. Computational cost $\varepsilon$-independent.

\[
u(t, x) = a(t, x)e^{i\phi(t, x)/\varepsilon} + O(\varepsilon).
\]

(#DOF and cost independent of $\varepsilon$)
The ansatz

\[ u(t, x) = a(t, x)e^{i\phi(t,x)/\varepsilon}, \]

generally breaks down in finite time if valid at \( t = 0 \).
Geometrical optics

- The ansatz
  \[ u(t, x) = a(t, x)e^{i\phi(t, x)}/\varepsilon, \]
  generally breaks down in finite time if valid at \( t = 0 \).
- Refraction of waves gives rise to multiple crossing waves
  \[ u(t, x) = \sum_{n=1}^{N} a_n(t, x)e^{i\phi_n(t, x)}/\varepsilon \]
  \( \Rightarrow \) Several amplitude and phase functions.
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Refraction of waves gives rise to multiple crossing waves

\[ u(t, x) = \sum_{n=1}^{N} a_n(t, x) e^{i \phi_n(t,x)/\varepsilon} \]

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Caustics appear at points of transition = concentration of rays.
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**Caustics** appear at points of transition = concentration of rays.

Geometrical optics predicts infinite amplitude at caustics.
Geometrical optics

- The ansatz

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- Refraction of waves gives rise to multiple crossing waves

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  ⇒ Several amplitude and phase functions.

- Caustics appear at points of transition = concentration of rays.

- Geometrical optics predicts infinite amplitude at caustics.

- Handling multiphase solutions tricky for numerical methods with fixed grids.
Caustics

Concentration of rays.

GO amplitude $a(t, y) \to \infty$ but should be $a(t, y) \sim \varepsilon^{-\alpha}$, $0 < \alpha < 1$.
Gaussian beams

- Approximate, localized, solutions to the wave equation/Schrodinger with a Gaussian profile (width $\sim \sqrt{\varepsilon}$).

Studied in e.g. Geophysics [Cerveny, Popov, Babich, Psencik, Klimes, Kravtsov, ...], Quantum Mechanics, [Heller, Hagedorn, Herman, Kluk, Kay, ...], Plasma Physics, [Pereverzev, Peeters, Maj, ...], Mathematics [Ralston, Hörmander, ...]. No breakdown at caustics.
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- No breakdown at caustics.
Gaussian beams

- Gaussian beams are of the same form as geometrical optics solutions,

\[ \psi(t, y) = A(t, y)e^{i\Phi(t, y)/\varepsilon}, \]

centered around a geometrical optics ray \( x(t) \),

\[ A(t, y) = a(t, y - x(t)), \quad \Phi(t, y) = \phi(t, y - x(t)). \]
Gaussian beams

- Gaussian beams are of the same form as geometrical optics solutions,
  \[ v(t, y) = A(t, y)e^{i\Phi(t, y)/\varepsilon}, \]
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- The phase \( \Phi \) will now have a positive imaginary part away from the ray \( x(t) \).
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- Imaginary part of \( \phi \sim |y|^2 \Rightarrow |\psi(t, y)| \sim e^{-|y-x(t)|^2/\varepsilon}, \)
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- Imaginary part of \( \phi \sim |y|^2 \Rightarrow |\mathbf{v}(t, y)| \sim e^{-|y-x(t)|^2/\varepsilon}, \)
  - Gaussian with width \( \sqrt{\varepsilon} \)
  - Localized around \( x(t) \). (Moves along the space time ray.)
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  - **Localized** around \( x(t) \). (Moves along the space time ray.)

- Phase \( \Phi(t, y) \) and amplitude \( A(t, y) \) approximated by polynomials locally around \( x(t) \)
Gaussian beams

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\[ \psi(t, y) = A(t, y) e^{i\Phi(t, y)}/\varepsilon, \]

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- The phase \( \Phi \) will now have a positive imaginary part away from the ray \( x(t) \).

- Imaginary part of \( \phi \sim |y|^2 \Rightarrow |\psi(t, y)| \sim \varepsilon e^{-|y-x(t)|^2}/\varepsilon \),

  - **Gaussian** with width \( \sqrt{\varepsilon} \)
  - **Localized** around \( x(t) \). (Moves along the space time ray.)

- Phase \( \Phi(t, y) \) and amplitude \( A(t, y) \) approximated by polynomials locally around \( x(t) \)

- \( \Phi(t, y) \) and \( A(t, y) \) solve eikonal and transport equation only up to \( O(|y - x|^m) \).
The simplest ("first order") Gaussian beams are of the form

\[ v(t, y) = a_0(t)e^{i \Phi(t,y)/\varepsilon}, \quad \Phi(t, y) = \phi(t, y - x(t)), \]

where

\[ \phi(t, y) = \phi_0(t) + y \cdot p(t) + \frac{1}{2} y \cdot M(t)y. \]

i.e. \( A(t, y) \) approximated to 0th order, and \( \Phi(t, y) \) to 2nd order.
First order beams

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i.e. \( A(t, y) \) approximated to 0th order, and \( \Phi(t, y) \) to 2nd order.

\[ \Rightarrow \]

We require that \( \Phi(t, y) \) solves eikonal to order \( O(|y - x|^3) \) and \( A(t, y) \) solves transport equation to order \( O(|y - x|) \).
First order beams

Let us thus require that

\[ \Phi_t^2 - c(y)^2|\nabla \Phi|^2 = O(|y - x(t)|^3), \]

\[ A_t + c\frac{\nabla \Phi \cdot \nabla A}{|\nabla \Phi|} + \frac{c^2 \Delta \Phi - \Phi_{tt}}{2c|\nabla \Phi|} A = O(|y - x(t)|), \]
First order beams

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\[ A_t + c \frac{\nabla \Phi \cdot \nabla A}{|\nabla \Phi|} + \frac{c^2 \Delta \Phi - \Phi_{tt}}{2c|\nabla \Phi|} A = O(|y - x(t)|), \]

\[ \Rightarrow \text{We obtain ODEs for } \phi_0, x, p, M, a_0. \]

\[ \dot{x}(t) = c(x)^2 p, \quad \dot{\phi}_0(t) = 0, \]
\[ \dot{p}(t) = -\nabla c(x)/c(x), \quad \dot{M}(t) = -D - MB - B^T M - MCM, \]
\[ \dot{a}_0(t) = \frac{a_0}{2} \left( -c(x)p \cdot \nabla c(x) - c(x)^3 p \cdot Mp + c(x)^2 \text{Tr}[M] \right), \]

where \( B, C, D \) are matrix functions involving \( x, p \) and \( c(x) \).
First order beams

Let us thus require that

\[
\Phi_t^2 - c(y)^2|\nabla \Phi|^2 = O(|y - x(t)|^3),
\]

\[
A_t + c \frac{\nabla \Phi \cdot \nabla A}{|\nabla \Phi|} + \frac{c^2 \Delta \Phi - \Phi_{tt}}{2c|\nabla \Phi|} A = O(|y - x(t)|),
\]

\[\Rightarrow\] We obtain ODEs for \( \phi_0, x, p, M, a_0. \)

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\begin{align*}
\dot{x}(t) &= c(x)^2 p , \\
\dot{\phi}_0(t) &= 0 , \\
\dot{p}(t) &= -\nabla c(x)/c(x) , \\
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\dot{a}_0(t) &= \frac{a_0}{2} \left( -c(x)p \cdot \nabla c(x) - c(x)^3 p \cdot Mp + c(x)^2 \text{Tr}[M] \right)
\end{align*}
\]

where \( B, C, D \) are matrix functions involving \( x, p \) and \( c(x) \).

- ODEs easy to solve numerically.
- Beams easy to evaluate:

\[
v(t, y) = a_0(t) e^{i\phi(t,y-x(t))/\varepsilon}, \quad \phi(t, y) = \phi_0(t) + y \cdot p(t) + \frac{1}{2} y \cdot M(t) y.
\]
First order beams

Let us thus require that

$$\Phi_t^2 - c(y)^2|\nabla \Phi|^2 = O(|y - x(t)|^3),$$

$$A_t + c \frac{\nabla \Phi \cdot \nabla A}{|\nabla \Phi|} + \frac{c^2 \Delta \Phi - \Phi_{tt}}{2c|\nabla \Phi|} A = O(|y - x(t)|),$$

⇒ We obtain ODEs for $\phi_0, x, p, M, a_0$.

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\begin{align*}
\dot{x}(t) &= c(x)^2 p, & \dot{\phi}_0(t) &= 0, \\
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\end{align*}
\]

⇒ Asymptotic order of accuracy is

$$\nu_{tt} - c(y)^2 \Delta \nu = O \left(\frac{1}{\sqrt{\varepsilon}}\right).$$
Higher order beams

More generally, we can construct higher order beams. Let

\[ \mathbf{v}(t, y) = a(t, y - x(t))e^{i\phi(t, y - x(t))/\varepsilon}, \]

where, for order \( K \) beams,
Higher order beams

More generally, we can construct higher order beams. Let

$$v(t, y) = a(t, y - x(t)) e^{i \phi(t, y-x(t))/\epsilon},$$

where, for order $K$ beams,

- The phase is a Taylor polynomial of order $K + 1$,

$$\phi(t, y) = \phi_0(t) + y \cdot p(t) + y \cdot \frac{1}{2} M(t)y + \sum_{|\beta|=3}^{K+1} \frac{1}{\beta!} \phi_\beta(t)y^\beta.$$
Higher order beams

More generally, we can construct higher order beams. Let

\[ \nu(t, y) = a(t, y - x(t)) e^{i \phi(t, y - x(t)) / \varepsilon}, \]

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- \( A \) is now a finite WKB expansion,

\[ a(t, y) = \sum_{j=0}^{[K/2]-1} \varepsilon^j a_j(t, y). \]
Higher order beams

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$$v(t, y) = a(t, y - x(t))e^{i\phi(t, y-x(t))/\varepsilon},$$

where, for order $K$ beams,

- The phase is a Taylor polynomial of order $K + 1$,

$$\phi(t, y) = \phi_0(t) + y \cdot p(t) + y \cdot \frac{1}{2} M(t)y + \sum_{|\beta|=3}^{K+1} \frac{1}{\beta!} \phi_{\beta}(t)y^\beta.$$ 

- $A$ is now a finite WKB expansion,

$$a(t, y) = \sum_{j=0}^{[K/2]-1} \varepsilon^j a_j(t, y)$$

- Each amplitude term $a_j$ is a Taylor polynomial to order $K - 2j - 1$

$$a_j(t, y) = \sum_{|\beta|=0}^{K-2j-1} \frac{1}{\beta!} a_{j, \beta}(t)y^\beta$$
Higher order beams

We now require that

- $\Phi(t, y) = \phi(t, y - x)$) solves eikonal equation to order $|y - x|^{K+2}$
- $a_j(t, y - x)$ solve higher order transport equations to order $|y - x|^{K-2j}$
Higher order beams

We now require that

- \( \Phi(t, y) = \phi(t, y - x) \) solves eikonal equation to order \( |y - x|^{K+2} \)
- \( a_j(t, y - x) \) solve higher order transport equations to order \( |y - x|^{K-2j} \)

Again, this gives ODEs for all Taylor coefficients,

\[
\begin{align*}
\dot{x}(t) &= c(x)^2 \rho, & \dot{\phi}_0(t) &= 0, \\
\dot{\rho}(t) &= -\nabla c(x)/c(x), & \dot{M}(t) &= -D - MB - B^T M - MCM, \\
\dot{a}_{j,\beta}(t) &= \ldots, & \dot{\phi}_\beta(t) &= \ldots,
\end{align*}
\]
Higher order beams

We now require that

- \( \Phi(t, y) = \phi(t, y - x) \) solves eikonal equation to order \(|y - x|^{K+2}\)
- \( a_j(t, y - x) \) solve higher order transport equations to order \(|y - x|^{K-2j}\)

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\dot{\rho}(t) &= -\nabla c(x)/c(x), \\
\dot{a}_j(t) &= \ldots, \\
\dot{\phi}_0(t) &= 0, \\
\dot{\phi}_\beta(t) &= \ldots,
\end{align*}
\]

Asymptotic order of accuracy is

\[
v_{tt} - c(y)^2 \Delta v = O\left(\varepsilon^{K/2-1}\right).
\]
\[ v(t, y) = a_0(t)e^{i\phi(t, y-x(t))/\varepsilon}, \quad \phi(t, y) = \phi_0(t) + y \cdot p(t) + \frac{1}{2} y \cdot M(t) y \]
Gaussian beams

Properties

\[ v(t, y) = a_0(t)e^{i\phi(t,y-x(t))/\varepsilon}, \quad \phi(t, y) = \phi_0(t) + y \cdot p(t) + \frac{1}{2} y \cdot M(t)y \]

- \( \Phi(t, x(t)) = \phi(t, 0) = \phi_0(t) \) is real valued
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Olof Runborg (KTH)

Gaussian Beam Approximation

IPP Garching, 2013
Gaussian beams

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- \( a_0(t) \) exists everywhere (no blow-up at caustics)
- Shape of beam remains Gaussian
- For high order need cutoff in a neighborhood of central ray to avoid spurious growth.

\[ v(t, y) = a(t, y - x(t))e^{i\phi(t, y-x(t))/\varepsilon} \rho(y - x(t)) \]
Superpositions of Gaussian beams

To approximate more general solutions, use superpositions of beams. Let \( v(t, y; z) \) be a beam starting from the point \( y = z \) and define

\[
u_{\text{GB}}(t, y) = \varepsilon - \frac{n}{2} \int_{K_0} v(t, y; z) \, dz
\]

\((n - \text{dimension, } K_0 - \text{compact set})\)
To approximate more general solutions, use superpositions of beams. Let \( \nu(t, y; z) \) be a beam starting from the point \( y = z \) and define

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For large values of \( \varepsilon \) it is sufficient to describe e.g. WKB data:

\[
\| A(y) e^{i \phi(y) / \varepsilon} - u_{GB}(0, \cdot) \|_E = O(\varepsilon^{-K_0/2}),
\]

Prefactor normalizes beams appropriately, \( \| u_{GB} \|_E = O(1) \).

By linearity of the wave equation equation a sum of solutions is also a solution.
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More general phase space superposition:

Let \( v(t, y; z, p) \) be a beam starting from the point \( y = z \) with momentum \( p \) and define

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u_{GB}(t, y) = \epsilon^{-n} \int_{\mathcal{K}_0} v(t, y; z, p) dz dp.
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\]

\( u_{GB}(t, y) \) is an asymptotic solution with initial data

\[
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Can describe more general data. (C.f. FBI transform.)
Approximate superposition integral by sum (trapezoidal rule)

\[ u_{GB}(t, y) = \varepsilon^{-\frac{n}{2}} \int_{K_0} v(t, y; z) dz \approx \varepsilon^{-\frac{n}{2}} \sum_{j} v(t, y; z_j) \Delta z^n. \]
Numerical methods

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- Lagrangian methods – Solve ODEs \( \forall z_j \) with standard methods. Similar to ray tracing but with all the additional Taylor coefficients computed along the rays \((M, a_j, \beta, \phi_\beta, \ldots)\) [Hill, Klimes, ...]
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- Eulerian methods – obtain parameters from solving PDEs on fixed grids [Leung, Qian, Burridge,07], [Jin, Wu, Yang,08], [Jin, Wu, Yang, Huang, 09], [Leung, Qian,09], [Qian,Ying,10],…

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- Wavefront methods – solve for parameters on a wave front [Motamed, OR,09]
Numerical methods

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Numerical issues

- Cost \(\sim\) number of beams since each beam is \(O(1)\).
  For accuracy need \(\Delta z \sim \sqrt{\varepsilon} \sim\) width of beams.
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  C.f. direct solution of wave equations, at least \(O(\varepsilon^{-(n+1)})\)
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  Wide beams \( \Rightarrow \) large Taylor approximation errors
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- Spreading of beams
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- Initial data approximation
  Many degrees of freedom. Can have huge impact on accuracy at later times.
Approximation errors

Let

\[ \Box := \partial_{tt} + c(y)^2 \Delta. \]

Suppose \( u \) is exact solution of wave equation and \( \tilde{u} \) is the Gaussian beam approximation

\[ \Box u = 0, \quad \Box \tilde{u} = O(\epsilon^{K/2-1}). \]

What is the norm error in \( \tilde{u} \), i.e. \( \| u - \tilde{u} \| \)?
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Use \( \varepsilon \)-scaled energy norm
\[ \| u \|_E^2 := \frac{\varepsilon^2}{2} \int_{\mathbb{R}^n} |u_t|^2 c(y)^{-2} + |\nabla u|^2 \, dy. \]

This is \( O(1) \) for WKB type initial data,
\[ u(0, x) = A(x) e^{i\phi(x)/\varepsilon} \Rightarrow \| u(0, \cdot) \|_E = O(1). \]
Approximation errors

Use well-posedness (stability) estimate for wave equation solutions $w$:

$$
\| w(t, \cdot) \|_E \leq \| w(0, \cdot) \|_E + \varepsilon C(T) \sup_{t \in [0, T]} \| \Box w(t, \cdot) \|_{L^2}, \quad 0 \leq t \leq T.
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Since, $\Box u = 0$ and by linearity

$$\Box[\tilde{u} - u] = \Box \tilde{u}.$$

Hence, assuming $u(0, x) = \tilde{u}(0, x)$,

$$\|\tilde{u}(t, \cdot) - u(t, \cdot)\|_E \leq \varepsilon C(T) \sup_{t \in [0, T]} \|\Box \tilde{u}(t, \cdot)\|_{L^2}, \quad 0 \leq t \leq T.$$

Error in $\tilde{u} \sim$ how well it satisfies equation, plus one order in $\varepsilon$
By earlier construction

\[ \square \tilde{u}_{GB}(t, x) = O(\varepsilon^{K/2 - 1}). \]

and

\[ \| \tilde{u}(t, \cdot) - u(t, \cdot) \|_E \leq \varepsilon C(T) \sup_{t \in [0, T]} \| \square \tilde{u}(t, \cdot) \|_{L^2}, \quad 0 \leq t \leq T. \]
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Hence, after appropriate normalization (so \( \|\tilde{u}\|_{E} = 1 \)) and assuming zero initial data error, [Ralston, 82]

\[ \sup_{t \in [0, T]} \|\tilde{u}(t, \cdot) - u(t, \cdot)\|_{E} \leq O(\varepsilon^{K/2}). \]

Estimate is sharp.
Norm estimates of $||u - u_{GB}||$ only rather recently derived [Swart, Rousse, Liu, Ralston, Tanushev, Bougacha, Alexandre, Lu, Yang,…]

Need to check how well $u_{GB}$ satisfies equation

$$\Box u_{GB} := \partial_{tt} u_{GB} + c(y)^2 \Delta u_{GB}, \quad u_{GB}(t, y) = \varepsilon^{-\frac{n}{2}} \int_{K_0} v(t, y; z) dz.$$
Approximation Errors Superpositions

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After scaling and Cauchy–Schwarz,

$$||\Box u_{GB}(y)||_2^2 \leq C\varepsilon^{-n} \int_{K_0} ||\Box v(t, y; z)||_2^2 dz \leq C\varepsilon^{-n} \varepsilon^{K-2+n/2}.$$
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- Gives error estimate for $u_{GB}$
  $$||u - u_{GB}||_E \leq \varepsilon C ||\Box u_{GB}||_2 \leq C \varepsilon^{K/2-n/4}.$$
Basic estimate

$$\|u(t, \cdot) - u_{GB}(t, \cdot)\|_E \leq O(\varepsilon^{K/2-n/4}).$$

is not sharp. E.g. it does not predict convergence for first order beams in 2D.
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- Gives a dependence on dimension \( n \) in estimate.
Theorem (Liu, Tanushev, O.R., 2010)

For the wave equation,

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For the Schrödinger equation,

\[ \| u(t, \cdot) - u_{GB}(t, \cdot) \|_{L^2} \leq O(\varepsilon^{K/2}) \, . \]
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- Result also for general scalar, strictly hyperbolic \( m \)-th order PDEs.
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- Result also for general scalar, strictly hyperbolic \( m \)-th order PDEs.
For time-harmonic waves consider Helmholtz equation

\[ \Delta u + (i\alpha \varepsilon^{-1} + \varepsilon^{-2}) n^2 u = g, \quad x \in \mathbb{R}^d. \]

where \( n(x) = 1/c(x) \), \( \alpha \) = damping and \( g \) supported on a co-dimension one manifold. (Ex. \( g = g_0(x_2)\delta(x_1)/\varepsilon \).)
Gaussian beams for Helmholtz

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- "Blobs" $\Rightarrow$ "Fat rays" localized around geometrical optics ray
- To leading order gaussian transversely to ray
Gaussian beams for Helmholtz

- Same ansatz,

\[ v = a(s, y - x(s)) e^{i \phi(s, y - x(s)) / \varepsilon}, \]

centered around a geometrical optics ray \( x(s) \) but \( s \) not time.
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- Similar ODEs for \( a_0, x, p, M, \phi_0 \) as in the time-dependent case.
Gaussian beams for Helmholtz

- Same ansatz,

\[ v = a(s, y - x(s))e^{i\phi(s, y - x(s))/\varepsilon}, \]

centered around a geometrical optics ray \( x(s) \) but \( s \) not time.

- First order beams are of the form

\[ \phi = \phi_0(s) + y \cdot p(s) + \frac{1}{2} y \cdot M(s)y, \quad a = a_0(s), \]

i.e. \( a \) approximated to 0th order, and \( \phi \) to 2nd order.

- Similar ODEs for \( a_0, x, p, M, \phi_0 \) as in the time-dependent case.

- Similar properties as in time-dependent case:
  - Phase \( \phi \) evaluated on ray = \( \phi_0(s) \) is real valued
  - If \( M(0) \) is symmetric and \( \Im M(0) \) is positive definite then this is true for \( M(s) \) (which exists) for all \( s > 0 \).
  - \( a_0(s) \) exists everywhere (no blow-up at caustics)
\[ v(y) = \]
\[ a(s, y - x(s)) e^{i\phi(s, y - x(s)) / \varepsilon}, \]

How to evaluate "\((s, y - x(s))\)" in expression for beam?
Gaussian beams for Helmholtz

Extension off ray

\[ v(y) = \]

\[ a(s^*, y - x(s^*)) e^{i\phi(s^*, y - x(s^*))}/\varepsilon, \]

\[ s^* = s^*(y) \]

- How to evaluate "\((s, y - x(s))\)" in expression for beam?
- No distinguished "time" variable ⇒ Extend beam by Taylor expansion transversely to ray:
  - Let \( s^* = s^*(y) \) such that \( x(s^*) \) is closest point on ray to \( y \).
Gaussian beams for Helmholtz

Extension off ray

\[
\nu(y) = a(s^*, y - x(s^*)) e^{i \phi(s^*, y - x(s^*))}/\varepsilon \times \varrho(y - x(s^*)) ,
\]
\[
s^* = s^*(y)
\]

- How to evaluate "\((s, y - x(s))" in expression for beam?"
- No distinguished "time" variable \( \Rightarrow \) Extend beam by Taylor expansion transversely to ray:
  - Let \( s^* = s^*(y) \) such that \( x(s^*) \) is closest point on ray to \( y \).
- Only well-defined close enough to ray \( \Rightarrow \) Cutoff \( \varrho(y) \) needed also for first order beams (size \( \eta \))
Helmholtz with source on $\Sigma = \{ y : \rho(y) = 0 \}$.

$$\Delta u + (i\alpha\varepsilon^{-1} + \varepsilon^{-2})n^2 u = \frac{1}{\varepsilon} g(y)\delta(\rho(y)).$$
Gaussian beams for Helmholtz

Source

Helmholtz with source on $\Sigma = \{ y : \rho(y) = 0 \}$.

$$\Delta u + (i\alpha \varepsilon^{-1} + \varepsilon^{-2})n^2 u = \frac{1}{\varepsilon} g(y) \delta(\rho(y)).$$

- Beams shoot out orthogonally in each direction from $\Sigma$
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- Gives beams $v^\pm(y)$, with $v^+(y) = 0$ when $\rho(y) < 0$ etc.
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$$Lu =: \Delta u + (i\alpha\varepsilon^{-1} + \varepsilon^{-2})n^2 u = \frac{1}{\varepsilon}g(y)\delta(\rho(y)).$$

• Beams shoot out orthogonally in each direction from $\Sigma$
• Gives beams $v^\pm(y)$, with $v^+(y) = 0$ when $\rho(y) < 0$ etc.
• Note that $v^+ = v^-$ on $\Sigma$, but $\nabla\phi^+ = -\nabla\phi^-$ so that $L(v^+ + v^-) \sim \delta(\rho(y)) + \text{smooth part.}$
\[ Lu =: \Delta u + (i\alpha\varepsilon^{-1} + \varepsilon^{-2})n^2 u \]
\[ = \frac{1}{\varepsilon} g(y)\delta(\rho(y)). \]

- Let \( v^\pm(y; z) \) be the beams starting from \( z \in \Sigma \) and define the superposition

\[
\begin{align*}
\sum_{x(s)} x_0 \\
\rho > 0 \\
\rho < 0 \\
u^+(x) \\
u^-(x) \\
\eta \\

u_{GB}(y) = \varepsilon^{-\frac{n-1}{2}} \int \sum_{y} [v^+(y; z) + v^-(y; z)] dA_z
\end{align*}
\]
Gaussian beams for Helmholtz

Superposition

\[ Lu =: \Delta u + (i\alpha\varepsilon^{-1} + \varepsilon^{-2})n^2 u \]
\[ = \frac{1}{\varepsilon} g(y) \delta(\rho(y)) . \]

- Let \( v^\pm(y; z) \) be the beams starting from \( z \in \Sigma \) and define superposition

\[ u_{GB}(y) = \varepsilon^{-\frac{n-1}{2}} \int_{\Sigma} [v^+(y; z) + v^-(y; z)] dA_z \]

(1)

- Choose initial data for beam \( v^\pm(z; z) \) such that

\[ Lu_{GB}(y) \sim \frac{1}{\varepsilon} \tilde{g}(y) \delta(\rho(y)) + f_{GB} \]

with \( \tilde{g} \approx g \).
Error estimate

Kth order beams, Helmholtz case

Theorem (Liu, Ralston, Tanushev, O.R., 2013)

Assume

- **Smooth, compactly supported source** $g(x)$
- **Index of refraction** $n(x)$ smooth and constant for $|x| > R$
- **No trapped rays**: $\exists L$ s.t. $|x(\pm L)| > 2R$ if $|x(0)| < R$, $|p(0)| = n(x(0))$
- **No initial data error** $\tilde{g} = g$

Then with $C$ independent of $\epsilon$ and $\alpha$,

$$\|u - u_{GB}\|_{L^2(|x|<R)} \leq C\epsilon^{K/2},$$
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Then with $C$ independent of $\varepsilon$ and $\alpha$,

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- Superposition in physical space.
- Convergence of all beams independent of dimension and presence of caustics.
Sketch of proof, wave equation

- Use energy estimate

\[ \|u_{GB}(t, \cdot) - u(t, \cdot)\|_E \leq \|u_{GB}(0, \cdot) - u(0, \cdot)\|_E + C\varepsilon \sup_{t \in [0, T]} \|\Box u_{GB}(t, \cdot)\|_{L^2}, \]

The residual is of the form

\[ \Box u_{GB}(t, y) = \varepsilon K/2 - q J \sum_{j=1}^{J} \varepsilon r_j T_{\varepsilon j}[f_j](t, y) + O(\varepsilon^\infty), \]

where \( r_j \geq 0, J \) finite and \( f_j \in L^2 \) (all independent of \( \varepsilon \)).

\( T_{\varepsilon j} : L^2 \rightarrow L^2 \) belongs to a class of oscillatory integral operators.

Together we get (if initial data exact)

\[ \|u_{GB}(t, \cdot) - u(t, \cdot)\|_E \leq C(T) \varepsilon K/2 J \sum_{j=1}^{J} \varepsilon r_j \|T_{\varepsilon j}\|_{L^2} |f_j|_{L^2} + O(\varepsilon^\infty). \]
Sketch of proof, wave equation

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- The residual is of the form

\[ \Box u_{GB}(t, y) = \varepsilon^{K/2-q} \sum_{j=1}^{J} \varepsilon^{r_j} T_j^\varepsilon[f_j](t, y) + O(\varepsilon^\infty), \]

where \( r_j \geq 0, J \text{ finite and } f_j \in L^2 \) (all independent of \( \varepsilon \)).

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We have

\[ \| u_{GB}(t, \cdot) - u(t, \cdot) \|_E \leq C(T) \varepsilon^{K/2} \sum_{j=1}^J \| T_j^\varepsilon \|_{L^2} + O(\varepsilon^\infty) \]

where, in its simplest form,

\[ T^\varepsilon[w](t, y) := \varepsilon^{-\frac{n+|\alpha|}{2}} \int_{K_0} w(z)(y - x(t; z))^\alpha e^{i\phi(t,y-x(t;z);z)/\varepsilon} dz, \]

for some multi-index \( \alpha \), Gaussian beam phase \( \phi \) and geometrical optics rays \( x(t; z) \) with \( x(0; z) = z \).
Sketch of proof, cont.

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for some multi-index $\alpha$, Gaussian beam phase $\phi$ and geometrical optics rays $x(t; z)$ with $x(0; z) = z$.

Result follows if we prove that $T^\varepsilon$ is bounded in $L^2$ independent of $\varepsilon$, $\|T^\varepsilon\|_{L^2} \leq C$.

This is the key estimate of our proof.
Sketch of proof, cont.

Estimate of $||T^\varepsilon||_{L^2}$, where

$$T^\varepsilon[w](t, y) := \varepsilon^{-\frac{n+|\alpha|}{2}} \int_{K_0} w(z)(y - x(t; z))^\alpha e^{i\phi(t, y - x(t; z); z)/\varepsilon} dz.$$ 

Main difficulty: no globally invertible map $x(0; z) = z \rightarrow x(t; z)$ because of caustics.
Sketch of proof, cont.

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- Main difficulty: no globally invertible map $x(0; z) = z \rightarrow x(t; z)$ because of caustics.
- Mapping $(x(0; z), p(0; z)) \rightarrow (x(t; z), p(t; z))$ is however globally invertible and smooth. Gives the "non-squeezing" property,

$$c_1|z - z'| \leq |p(t; z) - p(t; z')| + |x(t; z) - x(t; z')| \leq c_2|z - z'|.$$ 

Olof Runborg (KTH)
Gaussian Beam Approximation
IPP Garching, 2013
Sketch of proof, cont.

Estimate of $||\mathcal{T}^\varepsilon||_{L^2}$, where

$$
\mathcal{T}^\varepsilon[w](t, y) := \varepsilon^{-\frac{n+|\alpha|}{2}} \int_{K_0} w(z)(y - x(t; z))^\alpha e^{i\phi(t, y - x(t; z); z)/\varepsilon} dz.
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c_1|z - z'| \leq |p(t; z) - p(t; z')| + |x(t; z) - x(t; z')| \leq c_2|z - z'|.
$$

- Allows us to use stationary phase arguments close to caustics, and carefully control cancellations of oscillations there (similar to [Swart,Rousse], [Bougacha, Akian, Alexandre]).
The estimate
\[ \| u(t, \cdot) - u_{GB}(t, \cdot) \|_E \leq O(\varepsilon^{K/2}) \]
is sharp for individual beams (relative error). But for superpositions?

Predicts convergence rate of first order beam to be only \( O(\sqrt{\varepsilon}) \).

These beams are based on the same high frequency approximation as geometrical optics which has \( O(\varepsilon) \) accuracy.

Numerical experiments suggest a better rate for odd order beams.

For the Helmholtz case we have proved \([\text{Motamed, OR}]\) that
\[ |u(x) - u_{GB}(x)| \leq O(\varepsilon^\lceil K/2 \rceil) \]
for the Taylor expansion part of the error away from caustics. This gives \( O(\varepsilon) \) for first order beams.

More error cancellations coming in for odd order beams? (\( \Rightarrow \) no gain in using even order beams)
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Approximation errors

Remarks

The estimate
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is sharp for individual beams (relative error). But for superpositions?

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- More error cancellations coming in for odd order beams? ($\Rightarrow$ no gain in using even order beams)
Numerical examples

Cusp caustic

Consider the test case where

\[ \Phi(0, y) = -y_1 + y_2^2, \]

\[ A(0, y) = e^{-10|y|^2}. \]

- Cusp caustic at \( t = 0.5 \)
- Two fold caustics at \( t > 0.5 \)
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Numerical examples
Cusp caustic, convergence

Energy norm

\[ \| u_k(0.25, \cdot) - u_F(0.25, \cdot) \|_E \]

\[ \| u_k(0.75, \cdot) - u_F(0.75, \cdot) \|_E \]
Numerical examples
Cusp caustic, convergence

Max norm

\[ \|u_k(0.25, \cdot) - u_F(0.25, \cdot)\|_{L^\infty} \]

\[ \|u_k(0.75, \cdot) - u_F(0.75, \cdot)\|_{L^\infty} \]

- \( k=1 \)
- \( k=2 \)
- \( k=3 \)

Log-linear scale with values on the y-axis ranging from 10^{-3} to 10^{-1} and on the x-axis ranging from 1/1000 to 1/100.