Gaussian Beam Approximations of High Frequency Waves

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Joint with
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ESF Exploratory Workshop
IPP Garching, October 2013

High frequency waves

Cauchy problem for scalar wave equation

$$u_{tt} - c(\boldsymbol{x})^2 \Delta u = 0, \qquad (t, \boldsymbol{x}) \in \mathbb{R}^+ \times \mathbb{R}^n,$$

$$u(0, \boldsymbol{x}) = A(\boldsymbol{x}) e^{i\phi(\boldsymbol{x})/\varepsilon}, \qquad u_t(0, \boldsymbol{x}) = \frac{1}{\varepsilon} B(\boldsymbol{x}) e^{i\phi(\boldsymbol{x})/\varepsilon},$$

where $c(\boldsymbol{x})$ (variable) smooth speed of propagation.

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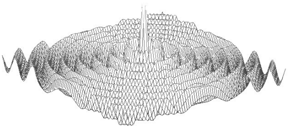
- High frequency $\rightarrow$
  short wave length $\rightarrow$
  highly oscillatory solutions $\rightarrow$
  many gridpoints.



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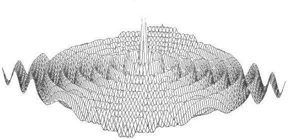

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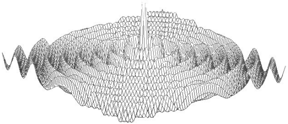
Solution u(x,y)



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- Direct numerical solution resolves wavelength:
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- Often unrealistic approach for applications in e.g. optics, electromagnetics, geophysics, acoustics, quantum mechanics, ...

High frequency material

Cauchy problem for scalar wave equation

$$u_{tt} - \nabla \cdot c^\varepsilon(\boldsymbol{x})\nabla u = 0, \qquad (t, \boldsymbol{x}) \in \mathbb{R}^+ \times \mathbb{R}^n,$$
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where $c^\varepsilon(\boldsymbol{x}) \in \mathbb{R}^{d\times d}$ has variations on length scale $\sim \varepsilon$.

The functions $A(\boldsymbol{x})$ and $B(\boldsymbol{x})$ are smooth (and independent of ε).

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  Prohibitively expensive when $\varepsilon \ll 1$, particularly in higher dimensions.
- Our approach: Heterogeneous Multiscale Method (HMM) [E,Engquist,2001].
- Solve small *micro* problems (localized in time and space) to probe effective dynamics, which is approximated on coarse grid ($\Delta x \gg \varepsilon$). Method cost (essentially) independent of ε.

Geometrical optics

Wave equation

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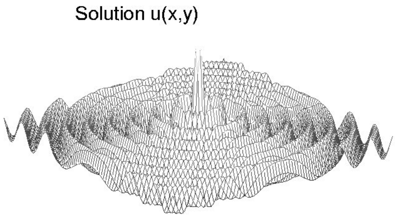
Write solution on the form

$$u(t, x) = a(t, x, \varepsilon) e^{i\phi(t,x)/\varepsilon}.$$



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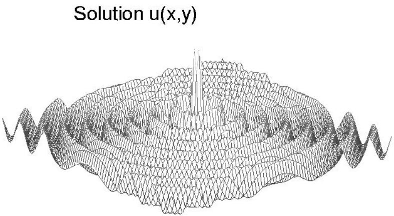

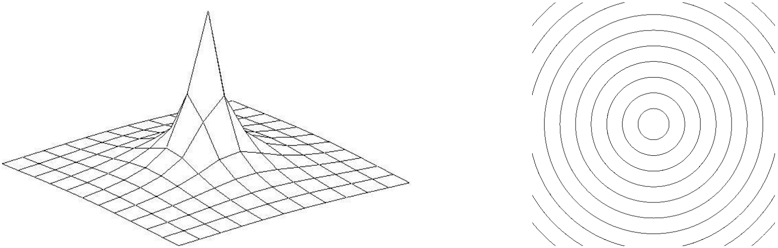

(a) Amplitude $a(x)$ (b) Phase $\phi(x)$

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- *a*, ϕ vary on a much coarser scale than *u*. (And varies little with ε.) Geometrical optics approximation considers *a* and ϕ as $\varepsilon \to 0$.

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  (And varies little with ε.) Geometrical optics approximation considers a and ϕ as $\varepsilon \to 0$.
- Phase and amplitude satisfy eikonal and transport equations

$$\phi_t^2 - c(y)^2|\nabla\phi|^2 = 0, \qquad a_t + c\frac{\nabla\phi\cdot\nabla a}{|\nabla\phi|} + \frac{c^2\Delta\phi - \phi_{tt}}{2c|\nabla\phi|}a = 0.$$

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- Ray tracing: $\boldsymbol{x}(t), \boldsymbol{p}(t)$ bicharacteristics of the eikonal equation,

$$\frac{d\boldsymbol{x}}{dt} = c(\boldsymbol{x})^2\boldsymbol{p}, \qquad \frac{d\boldsymbol{p}}{dt} = -\frac{\nabla c(\boldsymbol{x})}{c(\boldsymbol{x})}, \qquad \phi(t, \boldsymbol{x}(t)) = \phi(0, \boldsymbol{x}(0)).$$

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- Good accuracy for small ε. Computational cost ε-independent.

$$u(t,x) = a(t,x)e^{i\phi(t,\boldsymbol{x})/\varepsilon} + O(\varepsilon).$$

  (#DOF and cost independent of ε)

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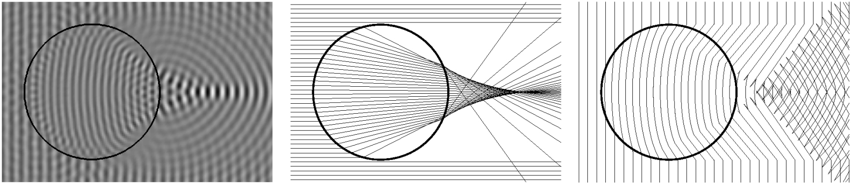
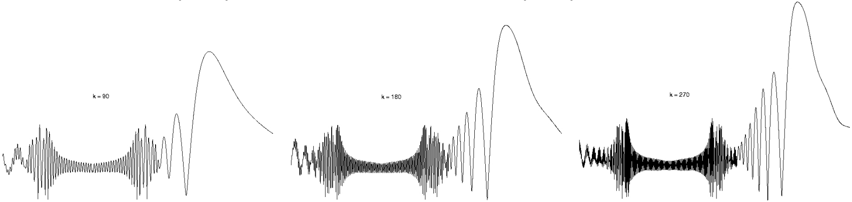
- Geometrical optics predicts infinite amplitude at caustics.
- Handling multiphase solutions tricky for numerical methods with fixed grids.

Caustics

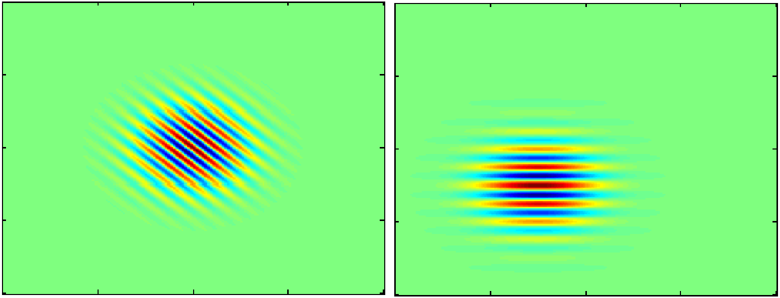
Concentration of rays.



GO amplitude $a(t, y) \to \infty$ but should be $a(t, y) \sim \varepsilon^{-\alpha}$, $0 < \alpha < 1$.



Gaussian beams



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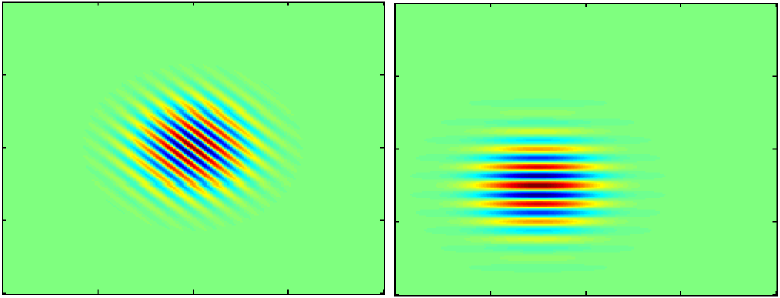

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- Studied in e.g. Geophysics [Cerveny, Popov, Babich, Psencik, Klimes, Kravtsov, ...], Quantum Mechanics,[Heller, Hagedorn, Herman, Kluk, Kay, ...], Plasma Physics, [Pereverzev, Peeters, Maj, ...], Mathematics [Ralston, Hörmander, ...]

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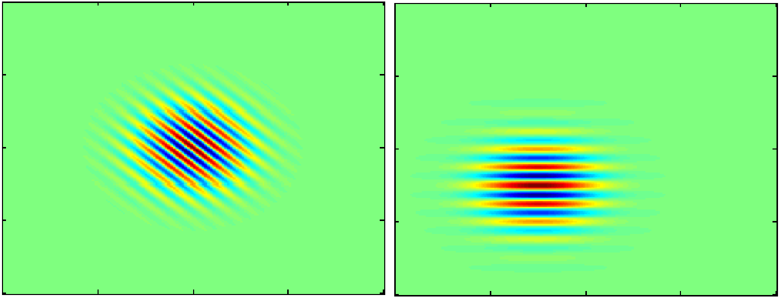

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- No breakdown at caustics.

Gaussian beams

- Gaussian beams are of the same form as geometrical optics solutions,
$$v(t,y) = A(t,y)e^{i\Phi(t,y)/\varepsilon},$$
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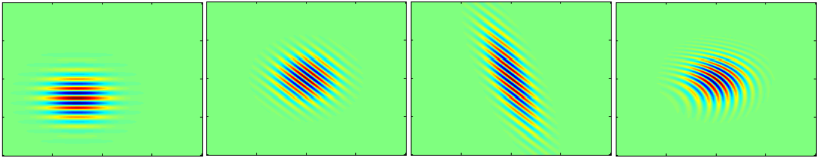
  - Localized around $x(t)$. (Moves along the space time ray.)
- Phase $\Phi(t,y)$ and amplitude $A(t,y)$ approximated by polynomials locally around $x(t)$
- $\Phi(t,y)$ and $A(t,y)$ solve eikonal and transport equation only upto $O(|y-x|^m)$.

First order beams



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$$v(t,y) = a_0(t)e^{i\Phi(t,y)/\varepsilon}, \qquad \Phi(t,y) = \phi(t, y - x(t)),$$

where

$$\phi(t,y) = \phi_0(t) + y \cdot p(t) + \frac{1}{2} y \cdot M(t) y.$$

i.e. $A(t,y)$ approximated to 0th order, and $\Phi(t,y)$ to 2nd order.

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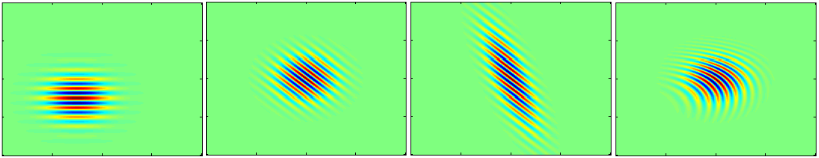

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$\Rightarrow$

We require that $\Phi(t,y)$ solves eikonal to order $O(|y-x|^3)$ and $A(t,y)$ solves transport equation to order $O(|y-x|)$.

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Let us thus require that

$$\Phi_t^2 - c(y)^2|\nabla\Phi|^2 = O(|y-x(t)|^3),$$

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$\Rightarrow$ We obtain ODEs for ϕ_0, x, p, M, a_0.

$$\dot{x}(t) = c(x)^2 p\,, \qquad \dot{\phi}_0(t) = 0\,,$$

$$\dot{p}(t) = -\nabla c(x)/c(x)\,, \qquad \dot{M}(t) = -D - MB - B^{\mathrm{T}}M - MCM\,,$$

$$\dot{a}_0(t) = \frac{a_0}{2}\left(-c(x)p\cdot\nabla c(x) - c(x)^3 p\cdot Mp + c(x)^2\mathrm{Tr}[M]\right)\,,$$

where B, C, D are matrix functions involving x, p and $c(x)$.

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- ODEs easy to solve numerically.
- Beams easy to evaluate:

$$v(t,y) = a_0(t)e^{i\phi(t,y-x(t))/\varepsilon}, \qquad \phi(t,y) = \phi_0(t) + y\cdot p(t) + \frac{1}{2}y\cdot M(t)y.$$

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$\Rightarrow$ Asymptotic order of accuracy is

$$v_{tt} - c(y)^2\Delta v = O\left(\frac{1}{\sqrt{\varepsilon}}\right).$$

Higher order beams

More generally, we can construct higher order beams. Let

$$v(t, y) = a(t, y - x(t))e^{i\phi(t, y - x(t))/\varepsilon},$$

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- The phase is a Taylor polynomial of order $K+1$,

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- A is now a finite WKB expansion,

$$a(t,y) = \sum_{j=0}^{\lceil K/2\rceil - 1} \varepsilon^j a_j(t,y)$$

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$$\phi(t, y) = \phi_0(t) + y \cdot p(t) + y \cdot \frac{1}{2}M(t)y + \sum_{|\beta|=3}^{K+1} \frac{1}{\beta!}\phi_\beta(t)y^\beta.$$

- A is now a finite WKB expansion,

$$a(t, y) = \sum_{j=0}^{\lceil K/2 \rceil - 1} \varepsilon^j a_j(t, y)$$

- Each amplitude term a_j is a Taylor polynomial to order $K - 2j - 1$

$$a_j(t, y) = \sum_{|\beta|=0}^{K-2j-1} \frac{1}{\beta!} a_{j,\beta}(t)y^\beta$$

Higher order beams

We now require that

- $\Phi(t, y) = \phi(t, y - x))$ solves eikonal equation to order $|y - x|^{K+2}$
- $a_j(t, y - x)$ solve higher order transport equations to order $|y - x|^{K-2j}$

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Again, this gives ODEs for all Taylor coefficients,

$$\begin{aligned}
\dot{x}(t) &= c(x)^2 p\,, & \dot{\phi}_0(t) &= 0\,, \\
\dot{p}(t) &= -\nabla c(x)/c(x)\,, & \dot{M}(t) &= -D - MB - B^T M - MCM\,, \\
\dot{a}_{j,\beta}(t) &= \ldots, & \dot{\phi}_\beta(t) &= \ldots,
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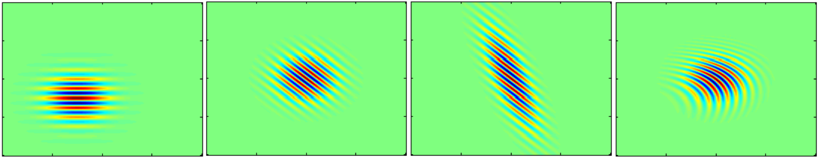
\end{aligned}
$$

Asymptotic order of accuracy is

$$
v_{tt} - c(y)^2 \Delta v = O\left(\varepsilon^{K/2-1}\right).
$$

Gaussian beams

Properties



$$v(t,y) = a_0(t)e^{i\phi(t,y-x(t))/\varepsilon}, \qquad \phi(t,y) = \phi_0(t) + y\cdot p(t) + \frac{1}{2}y\cdot M(t)y$$

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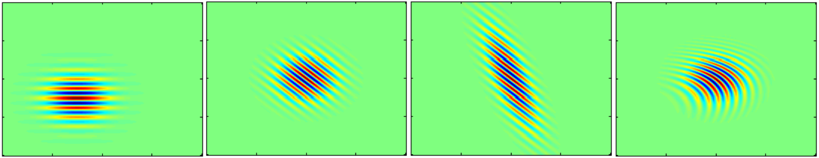

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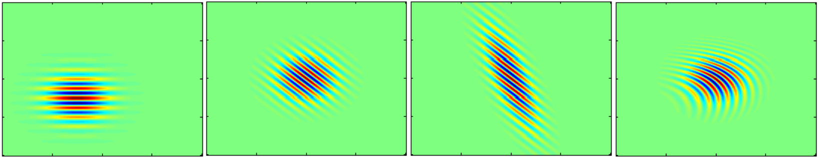

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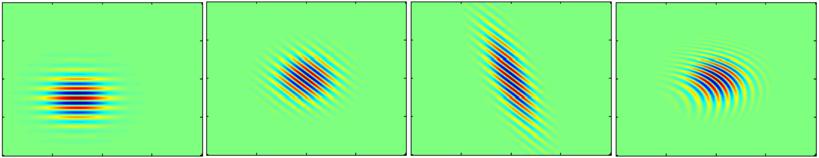

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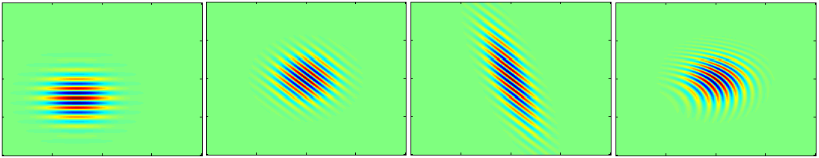

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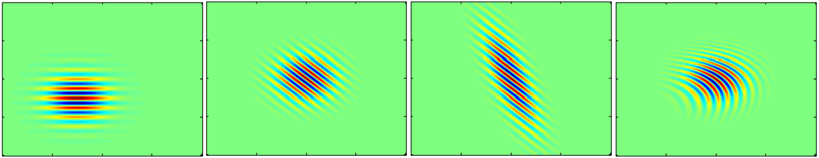

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- Shape of beam remains Gaussian
- For high order need cutoff in a neighborhood of central ray to avoid spurious growth.

$$v(t,y) = a(t,y-x(t))e^{i\phi(t,y-x(t))/\varepsilon}\varrho(y-x(t))$$

Superpositions of Gaussian beams

To approximate more general solutions, use superpositions of beams. Let $v(t, y; z)$ be a beam starting from the point $y = z$ and define

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(n – dimension, K_0 – compact set)

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  [Tanushev, 2007].
- Prefactor normalizes beams appropriately, $||u_{GB}||_E = O(1)$.

Superpositions of Gaussian beams

More general phase space superposition:

Let $v(t, y; z, p)$ be a beam starting from the point $y = z$ with momentum p and define

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Can describe more general data. (C.f. FBI transform.)

Numerical methods

- Approximate superposition integral by sum (trapezoidal rule)

$$u_{GB}(t,y) = \varepsilon^{-\frac{n}{2}} \int_{K_0} v(t,y;z)dz \approx \varepsilon^{-\frac{n}{2}} \sum_j v(t,y;z_j)\Delta z^n.$$

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- Cost $\sim$ number of beams since each beam is $O(1)$.
  For accuracy need $\Delta z \sim \sqrt{\varepsilon} \sim$ width of beams.
  $\Rightarrow$ cost $\sim O(\varepsilon^{-n/2})$
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- Spreading of beams
  Wide beams $\Rightarrow$ large Taylor approximation errors
- Initial data approximation
  Many degrees of freedom. Can have huge impact on accuracy at later times.

Approximation errors

Let

$$\Box := \partial_{tt} + c(y)^2\Delta.$$

Suppose u is exact solution of wave equation and $\tilde{u}$ is the Gaussian beam approximation

$$\Box u = 0, \qquad \Box\tilde{u} = O(\varepsilon^{K/2-1}).$$

What is the norm error in $\tilde{u}$, i.e. $||u - \tilde{u}||$?

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Use ε-scaled energy norm

$$||u||_E^2 := \frac{\varepsilon^2}{2}\int_{\mathbb{R}^n} |u_t|^2 c(y)^{-2} + |\nabla u|^2 dy \ .$$

This is $O(1)$ for WKB type initial data,

$$u(0,x) = A(x)e^{i\phi(x)/\varepsilon} \quad \Rightarrow \quad ||u(0,\cdot)||_E = O(1).$$

Approximation errors

Use well-posedness (stability) estimate for wave equation solutions w:

$$\|w(t,\cdot)\|_E \leq \|w(0,\cdot)\|_E + \varepsilon C(T) \sup_{t\in[0,T]} \|\Box w(t,\cdot)\|_{L^2}, \qquad 0 \leq t \leq T.$$

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Since, $\Box u = 0$ and by linearity

$$\Box[\tilde{u} - u] = \Box\tilde{u}.$$

Hence, assuming $u(0,x) = \tilde{u}(0,x)$,

$$\boxed{||\tilde{u}(t,\cdot) - u(t,\cdot)||_E \leq \varepsilon C(T) \sup_{t\in[0,T]} ||\Box\tilde{u}(t,\cdot)||_{L^2}, \qquad 0 \leq t \leq T.}$$

Error in $\tilde{u} \sim$ how well it satisfies equation, plus one order in ε

Approximation Errors Single Gaussian Beam

By earlier construction

$$\Box \tilde{u}_{GB}(t,x) = O(\varepsilon^{K/2-1}).$$

and

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Hence, after appropriate normalization (so $||\tilde{u}||_E = 1$) and assuming zero initial data error, [Ralston, 82]

$$\sup_{t\in[0,T]} ||\tilde{u}(t,\cdot) - u(t,\cdot)||_E \leq O(\varepsilon^{K/2}).$$

Estimate is sharp.

Approximation Errors Superpositions

Norm estimates of $||u - u_{GB}||$ only rather recently derived [Swart, Rousse, Liu, Ralston, Tanushev, Bougacha, Alexandre, Lu, Yang,...]

Need to check how well u_{GB} satisfies equation

$$\Box u_{GB} := \partial_{tt} u_{GB} + c(y)^2 \Delta u_{GB}, \qquad u_{GB}(t,y) = \varepsilon^{-\frac{n}{2}} \int_{K_0} v(t,y;z)dz.$$

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- After scaling and Cauchy–Schwarz,

$$||\Box u_{GB}(y)||_2^2 \leq C\varepsilon^{-n} \int_{K_0} ||\Box v(t,y;z)||_2^2 dz \leq C\varepsilon^{-n} \varepsilon^{K-2+n/2}.$$

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- Gives error estimate for u_{GB}

$$||u - u_{GB}||_E \leq \varepsilon C ||\Box u_{GB}||_2 \leq C\varepsilon^{K/2-n/4}.$$

Approximation errors superpositions

- Basic estimate

$$||u(t,\cdot) - u_{GB}(t,\cdot)||_E \leq O(\varepsilon^{K/2-n/4}) .$$

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- Gives a dependence on dimension n in estimate.

Error estimate

Theorem (Liu, Tanushev, O.R.,2010)

For the wave equation,

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- Result also for general scalar, strictly hyperbolic *m*-th order PDEs.

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- Superposition in physical space. Initial data approximated on a submanifold of phase space (WKB data).
- Convergence of all beams independent of dimension and presence of caustics.
- Result also for general scalar, strictly hyperbolic *m*-th order PDEs.
- Cf. [Bougacha, Akian, Alexandre, 2009], [Rousse, Swart, 2009] and [Lu, Yang, 2011] for other settings.

Gaussian beams for Helmholtz

For time-harmonic waves consider Helmholtz equation

$$\Delta u + (i\alpha\varepsilon^{-1} + \varepsilon^{-2})n^2 u = g, \qquad x \in \mathbb{R}^d.$$

where $n(x) = 1/c(x)$, α=damping and g supported on a co-dimension one manifold. (Ex. $g = g_0(x_2)\delta(x_1)/\varepsilon$.)

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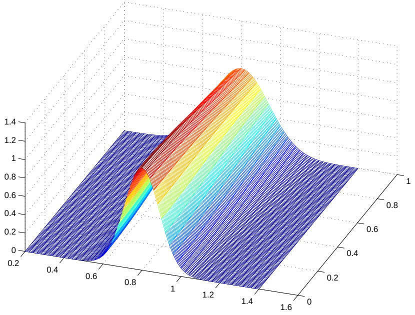

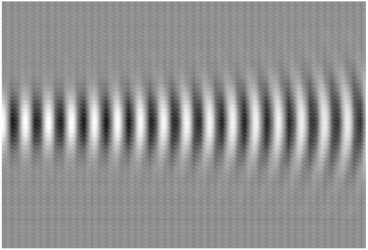

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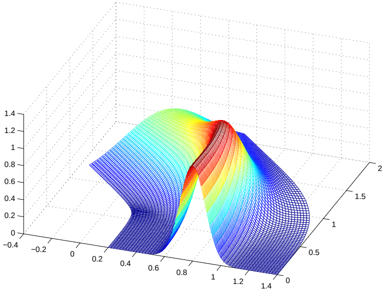

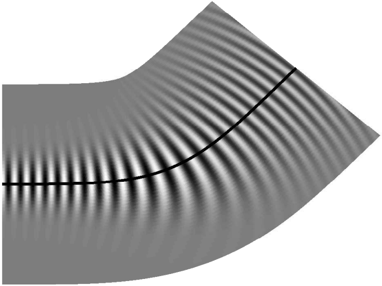

- "Blobs" $\Rightarrow$ "Fat rays" localized around geometrical optics ray
- To leading order gaussian transversely to ray

Gaussian beams for Helmholtz

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$$v = a(s, y - x(s))e^{i\phi(s,y-x(s))/\varepsilon},$$

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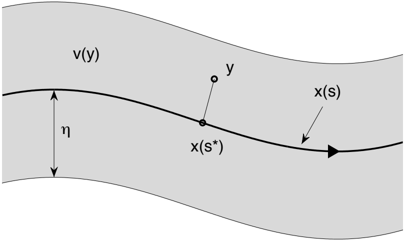
- Similar properties as in time-dependent case:
  - Phase ϕ evaluated on ray $= \phi_0(s)$ is real valued
  - If $M(0)$ is symmetric and $\Im M(0)$ is positive definite then this is true for $M(s)$ (which exists) for all $s > 0$.
  - $a_0(s)$ exists everywhere (no blow-up at caustics)

Gaussian beams for Helmholtz

Extension off ray

$$v(y) =$$

$$a(s, y - x(s))e^{i\phi(s,y-x(s))/\varepsilon},$$



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Gaussian beams for Helmholtz

Extension off ray

$$v(y) =$$

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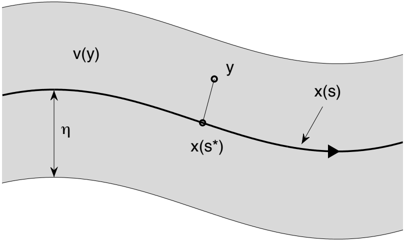

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Gaussian beams for Helmholtz

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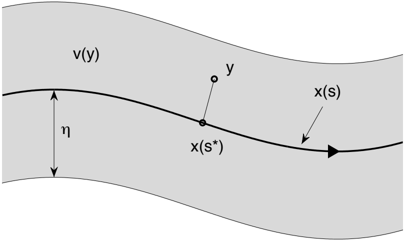

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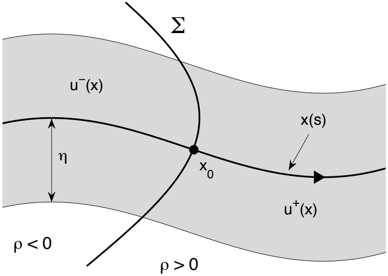
- No distinguished "time" variable $\Rightarrow$ Extend beam by Taylor expansion transversely to ray:
  Let $s^* = s^*(y)$ such that $x(s^*)$ is closest point on ray to y.
- Only well-defined close enough to ray $\Rightarrow$ Cutoff $\varrho(y)$ needed also for first order beams (size η)

Helmholtz with source on $\Sigma = \{y : \rho(y) = 0\}$.

$$\Delta u + (i\alpha\varepsilon^{-1} + \varepsilon^{-2})n^2 u = \frac{1}{\varepsilon} g(y)\delta(\rho(y)).$$



Gaussian beams for Helmholtz

Source

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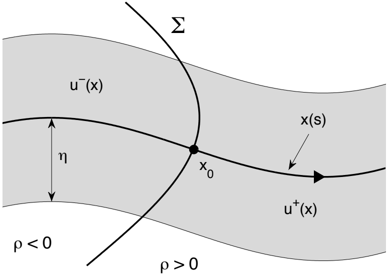

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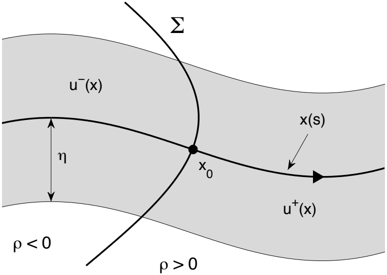

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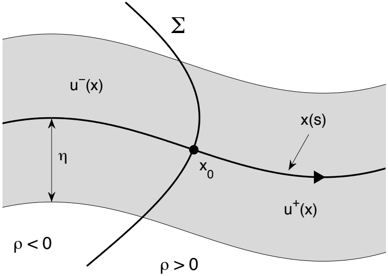
- Gives beams $v^{\pm}(y)$, with $v^{+}(y) = 0$ when $\rho(y) < 0$ etc.

Helmholtz with source on $\Sigma = \{y : \rho(y) = 0\}$.

$$\color{red}{Lu} =: \Delta u + (i\alpha\varepsilon^{-1} + \varepsilon^{-2})n^2 u = \frac{1}{\varepsilon} g(y)\delta(\rho(y)).$$



- Beams shoot out orthogonally in each direction from Σ
- Gives beams $v^{\pm}(y)$, with $v^{+}(y) = 0$ when $\rho(y) < 0$ etc.
- Note that $v^{+} = v^{-}$ on Σ, but $\nabla\phi^{+} = -\nabla\phi^{-}$ so that $L(v^{+} + v^{-}) \sim \delta(\rho(y)) +$ smooth part.

$$
\begin{aligned}
Lu &=: \Delta u + (i\alpha\varepsilon^{-1} + \varepsilon^{-2})n^2 u \\
&= \frac{1}{\varepsilon} g(y)\delta(\rho(y)).
\end{aligned}
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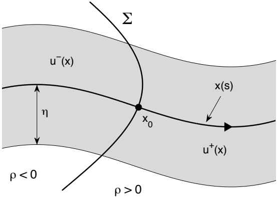

- Let $v^{\pm}(y; z)$ be the beams starting from $z \in \Sigma$ and define superposition

$$
u_{GB}(y) = \varepsilon^{-\frac{n-1}{2}} \int_{\Sigma} [v^{+}(y; z) + v^{-}(y; z)] dA_z \tag{1}
$$

Gaussian beams for Helmholtz

Superposition

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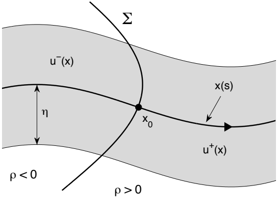

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- Choose initial data for beam $v^{\pm}(z; z)$ such that

$$Lu_{GB}(y) \sim \frac{1}{\varepsilon} \tilde{g}(y)\delta(\rho(y)) + f_{GB}$$

with $\tilde{g} \approx g$.

Error estimate

*K*th order beams, Helmholtz case

Theorem (Liu, Ralston, Tanushev, O.R., 2013)

Assume

- *Smooth, compactly supported source* $g(x)$
- *Index of refraction* $n(x)$ *smooth and constant for* $|x| > R$
- *No trapped rays:* $\exists L$ *s.t.* $|x(\pm L)| > 2R$ *if* $|x(0)| < R$, $|p(0)| = n(x(0))$
- *No initial data error* $\tilde{g} = g$.

Then with C *independent of* ε *and* α,

$$||u - u_{GB}||_{L^2(|x|<R)} \leq C\varepsilon^{K/2},$$

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Sketch of proof, wave equation

- Use energy estimate

$$\|u_{GB}(t,\cdot) - u(t,\cdot)\|_E \leq \|u_{GB}(0,\cdot) - u(0,\cdot)\|_E + C\varepsilon \sup_{t\in[0,T]} \|\Box u_{GB}(t,\cdot)\|_{L^2},$$

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- The residual is of the form

$$\Box u_{GB}(t,y) = \varepsilon^{K/2-q}\sum_{j=1}^{J} \varepsilon^{r_j}\mathcal{T}_j^{\varepsilon}[f_j](t,y) + \mathcal{O}(\varepsilon^{\infty}) \ ,$$

where $r_j \ge 0$, J finite and $f_j \in L^2$ (all independent of ε).
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- Together we get (if initial data exact)

$$\|u_{GB}(t,\cdot) - u(t,\cdot)\|_E \leq C(T)\varepsilon^{K/2} \sum_{j=1}^{J} \varepsilon^{r_j} \|\mathcal{T}_j^{\varepsilon}\|_{L^2} \, ||f_j||_{L^2} + \mathcal{O}(\varepsilon^{\infty})$$

Sketch of proof, cont.

We have

$$\|u_{GB}(t,\cdot) - u(t,\cdot)\|_E \leq C(T)\varepsilon^{K/2} \sum_{j=1}^{J} \|\mathcal{T}_j^\varepsilon\|_{L^2} + \mathcal{O}(\varepsilon^\infty)$$

where, in its simplest form,

$$\mathcal{T}^\varepsilon[w](t,y) := \varepsilon^{-\frac{n+|\alpha|}{2}} \int_{K_0} w(z)(y - x(t;z))^\alpha e^{i\phi(t,y-x(t;z);z)/\varepsilon} dz,$$

for some multi-index α, Gaussian beam phase ϕ and geometrical optics rays $x(t;z)$ with $x(0;z) = z$.

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Result follows if we prove that $\mathcal{T}^\varepsilon$ is bounded in L^2 independent of ε,

$$||\mathcal{T}^\varepsilon||_{L^2} \le C.$$

This is the key estimate of our proof.

Sketch of proof, cont.

Estimate of $||\mathcal{T}^\varepsilon||_{L^2}$, where

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$$c_1|z - z'| \leq |p(t;z) - p(t;z')| + |x(t;z) - x(t;z')| \leq c_2|z - z'| \; .$$

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- Allows us to use stationary phase arguments close to caustics, and carefully control cancellations of oscillations there (similar to [Swart,Rousse], [Bougacha, Akian, Alexandre]).

Approximation errors
Remarks

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- More error cancellations coming in for odd order beams? ($\Rightarrow$ no gain in using even order beams)

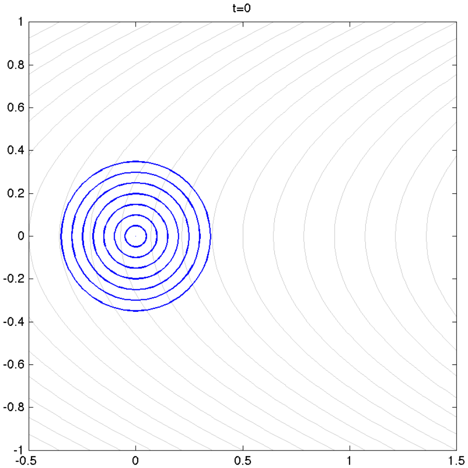
Numerical examples
Cusp caustic

Consider the test case where

$$\Phi(0, y) = -y_1 + y_2^2,$$

$$A(0, y) = e^{-10|y|^2}.$$

- Cusp caustic at $t = 0.5$
- Two fold caustics at $t > 0.5$



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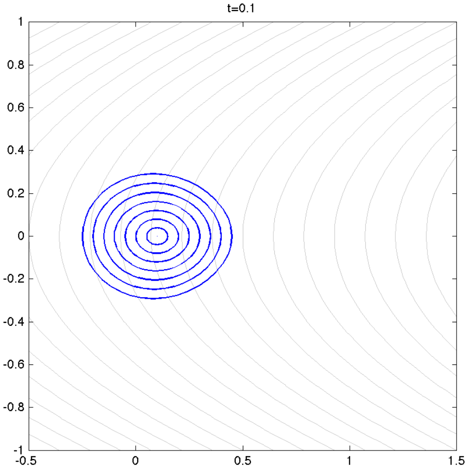

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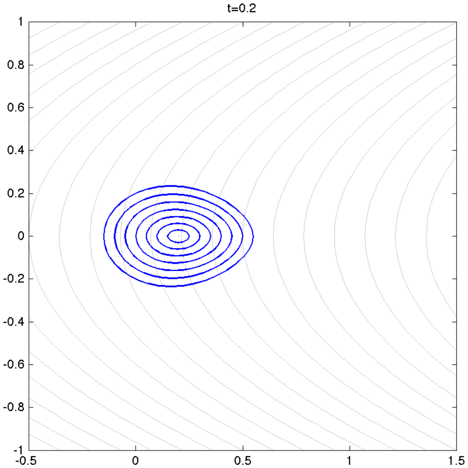

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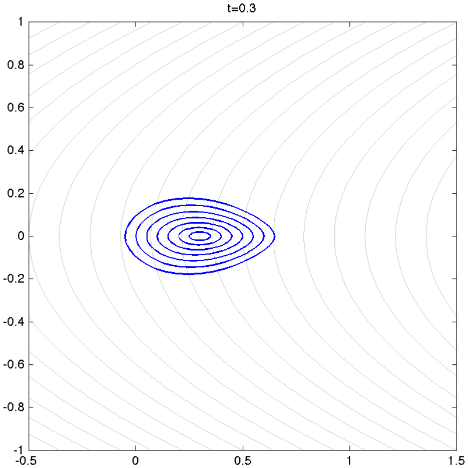

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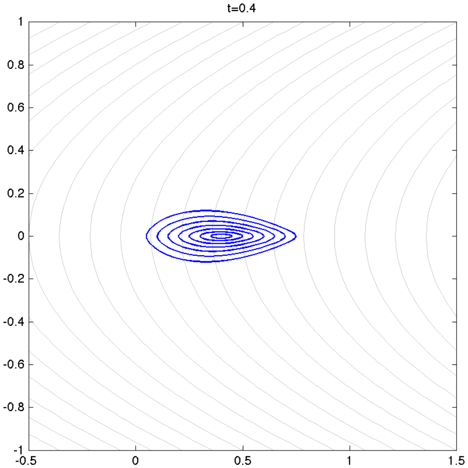

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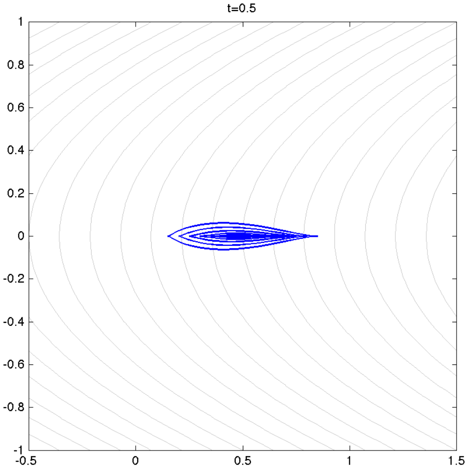

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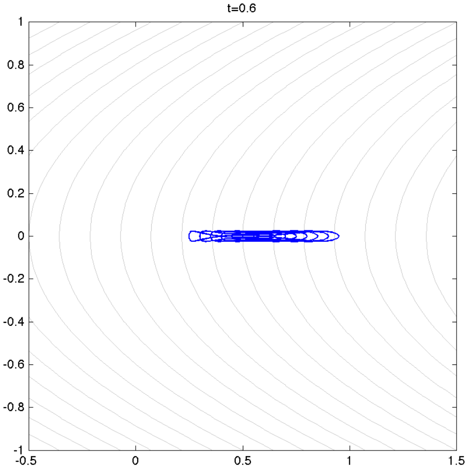

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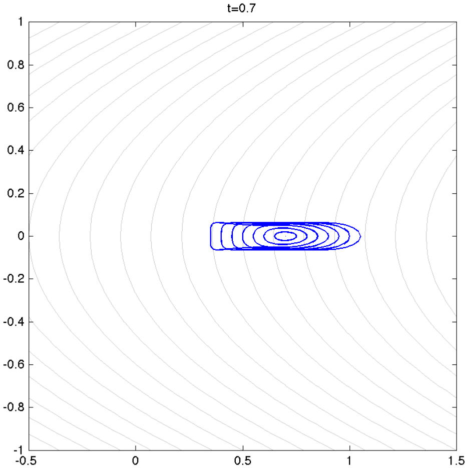

Consider the test case where

$$\Phi(0,y) = -y_1 + y_2^2,$$

$$A(0,y) = e^{-10|y|^2}.$$

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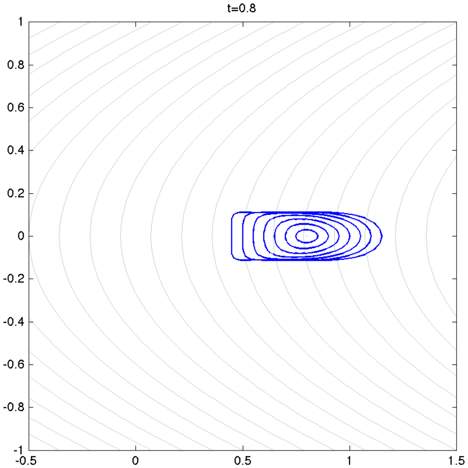

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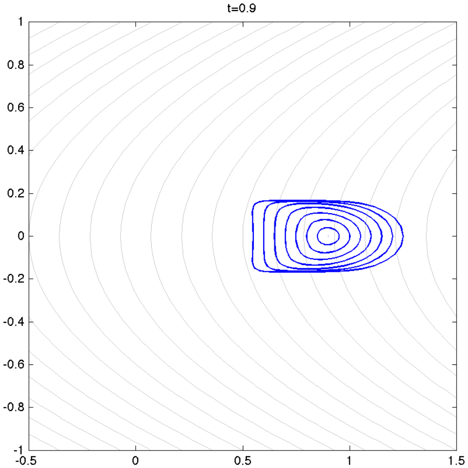

Numerical examples

Cusp caustic

Consider the test case where

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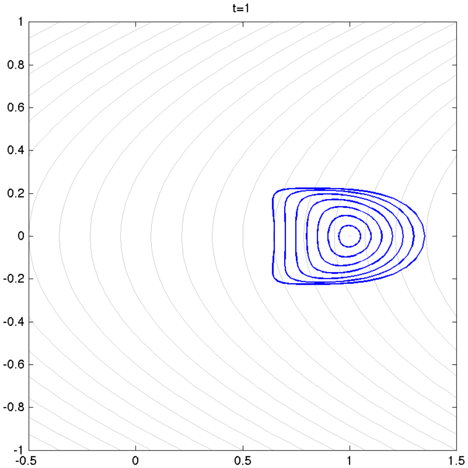
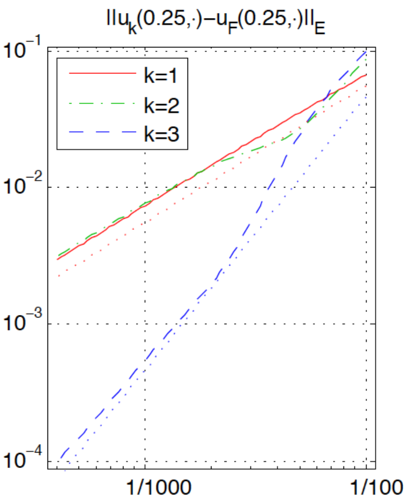


Numerical examples

Cusp caustic, convergence

Energy norm



Numerical examples

Cusp caustic, convergence

Max norm