Gaussian Beam Approximations of High Frequency Waves

Olof Runborg

Department of Mathematics, KTH

Joint with Hailiang Liu, Iowa, James Ralston, UCLA, Nick Tanushev, Z-Terra Corp.

ESF Exploratory Workshop IPP Garching, October 2013

Cauchy problem for scalar wave equation

$$u_{tt} - c(\mathbf{x})^2 \Delta u = 0, \qquad (t, \mathbf{x}) \in \mathbb{R}^+ \times \mathbb{R}^n,$$

$$u(0, \mathbf{x}) = A(\mathbf{x})e^{i\phi(\mathbf{x})/\varepsilon}, \qquad u_t(0, \mathbf{x}) = \frac{1}{\varepsilon}B(\mathbf{x})e^{i\phi(\mathbf{x})/\varepsilon},$$

where $c(\mathbf{x})$ (variable) smooth speed of propagation.

< ロ > < 同 > < 回 > < 回 >

Cauchy problem for scalar wave equation

$$u_{tt} - c(\mathbf{x})^2 \Delta u = 0, \qquad (t, \mathbf{x}) \in \mathbb{R}^+ \times \mathbb{R}^n,$$

 $u(0, \mathbf{x}) = A(\mathbf{x})e^{i\phi(\mathbf{x})/\varepsilon}, \qquad u_t(0, \mathbf{x}) = \frac{1}{\varepsilon}B(\mathbf{x})e^{i\phi(\mathbf{x})/\varepsilon},$

where $c(\mathbf{x})$ (variable) smooth speed of propagation.

 High frequency → short wave length → highly oscillatory solutions → many gridpoints.



Solution u(x,y)

Cauchy problem for scalar wave equation

$$u_{tt} - c(\mathbf{x})^2 \Delta u = 0, \qquad (t, \mathbf{x}) \in \mathbb{R}^+ \times \mathbb{R}^n,$$

 $u(0, \mathbf{x}) = A(\mathbf{x})e^{i\phi(\mathbf{x})/\varepsilon}, \qquad u_t(0, \mathbf{x}) = \frac{1}{\varepsilon}B(\mathbf{x})e^{i\phi(\mathbf{x})/\varepsilon},$

where $c(\mathbf{x})$ (variable) smooth speed of propagation.

 High frequency → short wave length → highly oscillatory solutions → many gridpoints.



Solution u(x,y)

 Direct numerical solution resolves wavelength: #gridpoints ~ ε⁻ⁿ at least ⇒ cost ~ ε⁻ⁿ⁻¹ at least

Cauchy problem for scalar wave equation

$$u_{tt} - c(\mathbf{x})^2 \Delta u = 0, \qquad (t, \mathbf{x}) \in \mathbb{R}^+ \times \mathbb{R}^n,$$

 $u(0, \mathbf{x}) = A(\mathbf{x})e^{i\phi(\mathbf{x})/\varepsilon}, \qquad u_t(0, \mathbf{x}) = \frac{1}{\varepsilon}B(\mathbf{x})e^{i\phi(\mathbf{x})/\varepsilon},$

where $c(\mathbf{x})$ (variable) smooth speed of propagation.

 High frequency → short wave length → highly oscillatory solutions → many gridpoints.



< ロ > < 同 > < 回 > < 回 >

Solution u(x,y)

- Direct numerical solution resolves wavelength: #gridpoints $\sim \varepsilon^{-n}$ at least $\Rightarrow \cos t \sim \varepsilon^{-n-1}$ at least
- Often unrealistic approach for applications in e.g. optics, electromagnetics, geophysics, acoustics, quantum mechanics, ...

Cauchy problem for scalar wave equation

$$egin{aligned} & u_{tt} -
abla \cdot oldsymbol{c}^arepsilon(oldsymbol{x})
abla u = 0, & (t,oldsymbol{x}) \in \mathbb{R}^+ imes \mathbb{R}^n, \ & u(0,oldsymbol{x}) = A(oldsymbol{x}), & u_t(0,oldsymbol{x}) = B(oldsymbol{x}), \end{aligned}$$

where $c^{\varepsilon}(\mathbf{x}) \in \mathbb{R}^{d \times d}$ has variations on length scale $\sim \varepsilon$.

The functions $A(\mathbf{x})$ and $B(\mathbf{x})$ are smooth (and independent of ε).

EN 4 EN

Cauchy problem for scalar wave equation

$$egin{aligned} & u_{tt} -
abla \cdot oldsymbol{c}^arepsilon(\mathbf{x})
abla u = 0, & (t, oldsymbol{x}) \in \mathbb{R}^+ imes \mathbb{R}^n, \ & u(0, oldsymbol{x}) = A(oldsymbol{x}), & u_t(0, oldsymbol{x}) = B(oldsymbol{x}), \end{aligned}$$

where $c^{\varepsilon}(\mathbf{x}) \in \mathbb{R}^{d \times d}$ has variations on length scale $\sim \varepsilon$.

The functions $A(\mathbf{x})$ and $B(\mathbf{x})$ are smooth (and independent of ε).

 Direct numerical solution resolves wavelength: #gridpoints ~ ε⁻ⁿ at least ⇒ cost ~ ε⁻ⁿ⁻¹ at least
 Prohibitively expensive when ε ≪ 1, particularly in higher dimensions.

Cauchy problem for scalar wave equation

$$egin{aligned} & u_{tt} -
abla \cdot oldsymbol{c}^arepsilon(\mathbf{x})
abla u = 0, & (t, oldsymbol{x}) \in \mathbb{R}^+ imes \mathbb{R}^n, \ & u(0, oldsymbol{x}) = A(oldsymbol{x}), & u_t(0, oldsymbol{x}) = B(oldsymbol{x}), \end{aligned}$$

where $c^{\varepsilon}(\mathbf{x}) \in \mathbb{R}^{d \times d}$ has variations on length scale $\sim \varepsilon$.

The functions $A(\mathbf{x})$ and $B(\mathbf{x})$ are smooth (and independent of ε).

- Direct numerical solution resolves wavelength: #gridpoints ~ ε⁻ⁿ at least ⇒ cost ~ ε⁻ⁿ⁻¹ at least
 Prohibitively expensive when ε ≪ 1, particularly in higher dimensions.
- Our approach: Heterogeneous Multiscale Method (HMM) [E,Engquist,2001].

3

Cauchy problem for scalar wave equation

$$\begin{split} u_{tt} - \nabla \cdot \boldsymbol{c}^{\varepsilon}(\boldsymbol{x}) \nabla u &= 0, \qquad (t, \boldsymbol{x}) \in \mathbb{R}^{+} \times \mathbb{R}^{n}, \\ u(0, \boldsymbol{x}) &= \boldsymbol{A}(\boldsymbol{x}), \qquad u_{t}(0, \boldsymbol{x}) = \boldsymbol{B}(\boldsymbol{x}), \end{split}$$

where $c^{\varepsilon}(\mathbf{x}) \in \mathbb{R}^{d \times d}$ has variations on length scale $\sim \varepsilon$.

The functions $A(\mathbf{x})$ and $B(\mathbf{x})$ are smooth (and independent of ε).

- Direct numerical solution resolves wavelength: #gridpoints ~ ε⁻ⁿ at least ⇒ cost ~ ε⁻ⁿ⁻¹ at least
 Prohibitively expensive when ε ≪ 1, particularly in higher dimensions.
- Our approach: Heterogeneous Multiscale Method (HMM) [E,Engquist,2001].

Solve small *micro* problems (localized in time and space) to probe effective dynamics, which is approximated on coarse grid (Δx ≫ ε). Method cost (essentially) independent of ε.

Wave equation

$$u_{tt}-c(\mathbf{x})^2\Delta u=0.$$

Write solution on the form

$$u(t,x) = a(t,x,\varepsilon)e^{i\phi(t,x)/\varepsilon}$$



Wave equation

$$u_{tt} - c(\mathbf{x})^2 \Delta u = 0.$$

Write solution on the form

$$u(t,x) = a(t,x,\varepsilon)e^{i\phi(t,x)/\varepsilon}$$





Olof Runborg (KTH)

a, φ vary on a much coarser scale than u.
 (And varies little with ε.) Geometrical optics approximation considers a and φ as ε → 0.

- a, φ vary on a much coarser scale than u.
 (And varies little with ε.) Geometrical optics approximation considers a and φ as ε → 0.
- Phase and amplitude satisfy eikonal and transport equations

$$\phi_t^2 - c(y)^2 |\nabla \phi|^2 = 0, \qquad a_t + c rac{
abla \phi \cdot
abla a}{|
abla \phi|} + rac{c^2 \Delta \phi - \phi_{tt}}{2c |
abla \phi|} a = 0.$$

< 47 ▶

- a, φ vary on a much coarser scale than u.
 (And varies little with ε.) Geometrical optics approximation considers a and φ as ε → 0.
- Phase and amplitude satisfy eikonal and transport equations

$$\phi_t^2 - c(y)^2 |\nabla \phi|^2 = 0, \qquad a_t + c \frac{\nabla \phi \cdot \nabla a}{|\nabla \phi|} + \frac{c^2 \Delta \phi - \phi_{tt}}{2c |\nabla \phi|} a = 0.$$

• Ray tracing: $\mathbf{x}(t)$, $\mathbf{p}(t)$ bicharacteristics of the eikonal equation,

$$\frac{d\boldsymbol{x}}{dt} = c(\boldsymbol{x})^2 \boldsymbol{p}, \qquad \frac{d\boldsymbol{p}}{dt} = -\frac{\nabla c(\boldsymbol{x})}{c(\boldsymbol{x})}, \qquad \phi(t, \boldsymbol{x}(t)) = \phi(0, \boldsymbol{x}(0)).$$

3 > 4 3

A D M A A A M M

- a, φ vary on a much coarser scale than u.
 (And varies little with ε.) Geometrical optics approximation considers a and φ as ε → 0.
- Phase and amplitude satisfy eikonal and transport equations

$$\phi_t^2 - c(y)^2 |\nabla \phi|^2 = 0, \qquad a_t + c rac{
abla \phi \cdot
abla a}{|
abla \phi|} + rac{c^2 \Delta \phi - \phi_{tt}}{2c |
abla \phi|} a = 0.$$

• Ray tracing: $\mathbf{x}(t)$, $\mathbf{p}(t)$ bicharacteristics of the eikonal equation,

$$\frac{d\boldsymbol{x}}{dt} = c(\boldsymbol{x})^2 \boldsymbol{p}, \qquad \frac{d\boldsymbol{p}}{dt} = -\frac{\nabla c(\boldsymbol{x})}{c(\boldsymbol{x})}, \qquad \phi(t, \boldsymbol{x}(t)) = \phi(0, \boldsymbol{x}(0)).$$

• Good accuracy for small ε . Computational cost ε -independent.

$$u(t,x) = a(t,x)e^{i\phi(t,x)/\varepsilon} + O(\varepsilon).$$

(#DOF and cost independent of ε)

< ロ > < 同 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ >

$$u(t,x) = a(t,x)e^{i\phi(t,x)/\varepsilon},$$

generally breaks down in finite time if valid at t = 0.

< ロ > < 同 > < 回 > < 回 >

$$u(t,x) = a(t,x)e^{i\phi(t,x)/\varepsilon},$$

generally breaks down in finite time if valid at t = 0.

• Refraction of waves gives rise to multiple crossing waves

$$u(t,x) = \sum_{n=1}^{N} a_n(t,x) e^{i\phi_n(t,x)/\varepsilon}$$

 \Rightarrow Several amplitude and phase functions.

3 > 4 3

$$u(t,x) = a(t,x)e^{i\phi(t,x)/\varepsilon},$$

generally breaks down in finite time if valid at t = 0.

Refraction of waves gives rise to multiple crossing waves

$$u(t,x) = \sum_{n=1}^{N} a_n(t,x) e^{i\phi_n(t,x)/\varepsilon}$$

- \Rightarrow Several amplitude and phase functions.
- Caustics appear at points of transition = concentration of rays.

3 > 4 3

< 47 ▶

$$u(t,x) = a(t,x)e^{i\phi(t,x)/\varepsilon},$$

generally breaks down in finite time if valid at t = 0.

• Refraction of waves gives rise to multiple crossing waves

$$u(t,x) = \sum_{n=1}^{N} a_n(t,x) e^{i\phi_n(t,x)/\varepsilon}$$

- \Rightarrow Several amplitude and phase functions.
- Caustics appear at points of transition = concentration of rays.
- Geometrical optics predicts infinite amplitude at caustics.

3 > 4 3

< 47 ▶

$$u(t,x) = a(t,x)e^{i\phi(t,x)/\varepsilon},$$

generally breaks down in finite time if valid at t = 0.

• Refraction of waves gives rise to multiple crossing waves

$$u(t,x) = \sum_{n=1}^{N} a_n(t,x) e^{i\phi_n(t,x)/\varepsilon}$$

- \Rightarrow Several amplitude and phase functions.
- Caustics appear at points of transition = concentration of rays.
- Geometrical optics predicts infinite amplitude at caustics.
- Handling multiphase solutions tricky for numerical methods with fixed grids.

< ロ > < 同 > < 回 > < 回 >

Caustics

Concentration of rays.



GO amplitude $a(t, y) \rightarrow \infty$ but should be $a(t, y) \sim \varepsilon^{-\alpha}$, $0 < \alpha < 1$.



< 47 ▶

H 5



 Approximate, localized, solutions to the wave equation/Schrodinger with a Gaussian profile (width ~ √ε).

< 47 ▶



- Approximate, localized, solutions to the wave equation/Schrodinger with a Gaussian profile (width $\sim \sqrt{\varepsilon}$).
- Studied in e.g. Geophysics [Cerveny, Popov, Babich, Psencik, Klimes, Kravtsov, ...], Quantum Mechanics, [Heller, Hagedorn, Herman, Kluk, Kay, ...], Plasma Physics, [Pereverzev, Peeters, Maj, ...], Mathematics [Ralston, Hörmander, ...]



- Approximate, localized, solutions to the wave equation/Schrodinger with a Gaussian profile (width $\sim \sqrt{\varepsilon}$).
- Studied in e.g. Geophysics [Cerveny, Popov, Babich, Psencik, Klimes, Kravtsov, ...], Quantum Mechanics, [Heller, Hagedorn, Herman, Kluk, Kay, ...], Plasma Physics, [Pereverzev, Peeters, Maj, ...], Mathematics [Ralston, Hörmander, ...]
- No breakdown at caustics.

Gaussian beams are of the same form as geometrical optics solutions,

$$\mathbf{v}(t,\mathbf{y})=\mathbf{A}(t,\mathbf{y})\mathbf{e}^{i\Phi(t,\mathbf{y})/\varepsilon},$$

centered around a geometrical optics ray x(t),

$$A(t, y) = a(t, y - x(t)), \qquad \Phi(t, y) = \phi(t, y - x(t)).$$

A D M A A A M M

Gaussian beams are of the same form as geometrical optics solutions,

$$\mathbf{v}(t,\mathbf{y})=\mathbf{A}(t,\mathbf{y})\mathbf{e}^{i\Phi(t,\mathbf{y})/\varepsilon},$$

centered around a geometrical optics ray x(t),

$$A(t, y) = a(t, y - x(t)), \qquad \Phi(t, y) = \phi(t, y - x(t)).$$

 The phase Φ will now have a positive imaginary part away from the ray x(t).

Gaussian beams are of the same form as geometrical optics solutions,

$$\mathbf{v}(t,\mathbf{y}) = \mathbf{A}(t,\mathbf{y})\mathbf{e}^{i\Phi(t,\mathbf{y})/\varepsilon},$$

centered around a geometrical optics ray x(t),

$$A(t, y) = a(t, y - x(t)), \qquad \Phi(t, y) = \phi(t, y - x(t)).$$

- The phase Φ will now have a positive imaginary part away from the ray x(t).
- Imaginary part of $\phi \sim |y|^2 \Rightarrow |v(t,y)| \sim e^{-|y-x(t)|^2/\varepsilon}$,

Gaussian beams are of the same form as geometrical optics solutions,

$$\mathbf{v}(t,\mathbf{y}) = \mathbf{A}(t,\mathbf{y})\mathbf{e}^{i\Phi(t,\mathbf{y})/\varepsilon}$$

centered around a geometrical optics ray x(t),

$$A(t, y) = a(t, y - x(t)), \qquad \Phi(t, y) = \phi(t, y - x(t)).$$

- The phase Φ will now have a positive imaginary part away from the ray x(t).
- Imaginary part of $\phi \sim |y|^2 \Rightarrow |v(t,y)| \sim e^{-|y-x(t)|^2/\varepsilon}$,
 - Gaussian with width $\sqrt{\varepsilon}$
 - Localized around x(t). (Moves along the space time ray.)

A B A B A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A

• Gaussian beams are of the same form as geometrical optics solutions,

$$\mathbf{v}(t,\mathbf{y}) = \mathbf{A}(t,\mathbf{y})\mathbf{e}^{i\Phi(t,\mathbf{y})/\varepsilon}$$

centered around a geometrical optics ray x(t),

$$A(t, y) = a(t, y - x(t)), \qquad \Phi(t, y) = \phi(t, y - x(t)).$$

- The phase Φ will now have a positive imaginary part away from the ray x(t).
- Imaginary part of $\phi \sim |y|^2 \Rightarrow |v(t,y)| \sim e^{-|y-x(t)|^2/\varepsilon}$,
 - Gaussian with width $\sqrt{\varepsilon}$
 - Localized around x(t). (Moves along the space time ray.)
- Phase Φ(t, y) and amplitude A(t, y) approximated by polynomials locally around x(t)

• Gaussian beams are of the same form as geometrical optics solutions,

$$\mathbf{v}(t,\mathbf{y}) = \mathbf{A}(t,\mathbf{y})\mathbf{e}^{i\Phi(t,\mathbf{y})/\varepsilon}$$

centered around a geometrical optics ray x(t),

$$A(t, y) = a(t, y - x(t)), \qquad \Phi(t, y) = \phi(t, y - x(t)).$$

- The phase Φ will now have a positive imaginary part away from the ray x(t).
- Imaginary part of $\phi \sim |y|^2 \Rightarrow |v(t,y)| \sim e^{-|y-x(t)|^2/\varepsilon}$,
 - Gaussian with width $\sqrt{\varepsilon}$
 - Localized around x(t). (Moves along the space time ray.)
- Phase Φ(t, y) and amplitude A(t, y) approximated by polynomials locally around x(t)
- $\Phi(t, y)$ and A(t, y) solve eikonal and transport equation only upto $O(|y x|^m)$.



The simplest ("first order") Gaussian beams are of the form

$$\mathbf{v}(t,\mathbf{y}) = \mathbf{a}_0(t)\mathbf{e}^{i\Phi(t,\mathbf{y})/\varepsilon}, \qquad \Phi(t,\mathbf{y}) = \phi(t,\mathbf{y}-\mathbf{x}(t)),$$

where

$$\phi(t,y) = \phi_0(t) + y \cdot \rho(t) + \frac{1}{2}y \cdot M(t)y.$$

i.e. A(t, y) approximated to 0th order, and $\Phi(t, y)$ to 2nd order.

< ロ > < 同 > < 回 > < 回 >



The simplest ("first order") Gaussian beams are of the form

$$\mathbf{v}(t,\mathbf{y}) = \mathbf{a}_0(t)\mathbf{e}^{i\Phi(t,\mathbf{y})/\varepsilon}, \qquad \Phi(t,\mathbf{y}) = \phi(t,\mathbf{y}-\mathbf{x}(t)),$$

where

 \Rightarrow

$$\phi(t, y) = \phi_0(t) + y \cdot p(t) + \frac{1}{2}y \cdot M(t)y.$$

i.e. A(t, y) approximated to 0th order, and $\Phi(t, y)$ to 2nd order.

We require that $\Phi(t, y)$ solves eikonal to order $O(|y - x|^3)$ and A(t, y) solves transport equation to order O(|y - x|).

Olof Runborg (KTH)

Let us thus require that

$$\Phi_t^2 - c(y)^2 |\nabla \Phi|^2 = O(|y - x(t)|^3),$$

 $A_t + c rac{\nabla \Phi \cdot \nabla A}{|\nabla \Phi|} + rac{c^2 \Delta \Phi - \Phi_{tt}}{2c |\nabla \Phi|} A = O(|y - x(t)|),$

2

Let us thus require that

$$\Phi_t^2 - c(y)^2 |\nabla \Phi|^2 = O(|y - x(t)|^3),$$

 $A_t + c rac{\nabla \Phi \cdot \nabla A}{|\nabla \Phi|} + rac{c^2 \Delta \Phi - \Phi_{tt}}{2c |\nabla \Phi|} A = O(|y - x(t)|),$

 \Rightarrow We obtain ODEs for ϕ_0 , x, p, M, a_0 .

$$\begin{split} \dot{x}(t) &= c(x)^2 p , \qquad \dot{\phi}_0(t) = 0 , \\ \dot{p}(t) &= -\nabla c(x)/c(x) , \qquad \dot{M}(t) = -D - MB - B^{\mathsf{T}}M - MCM , \\ \dot{a}_0(t) &= \frac{a_0}{2} \left(-c(x)p \cdot \nabla c(x) - c(x)^3 p \cdot Mp + c(x)^2 \mathrm{Tr}[M] \right) , \end{split}$$

where *B*, *C*, *D* are matrix functions involving x, p and c(x).

Let us thus require that

$$\Phi_t^2 - c(y)^2 |\nabla \Phi|^2 = O(|y - x(t)|^3),$$

 $A_t + c \frac{\nabla \Phi \cdot \nabla A}{|\nabla \Phi|} + \frac{c^2 \Delta \Phi - \Phi_{tt}}{2c |\nabla \Phi|} A = O(|y - x(t)|),$

 \Rightarrow We obtain ODEs for ϕ_0 , x, p, M, a_0 .

$$\begin{split} \dot{x}(t) &= c(x)^2 p , \qquad \dot{\phi}_0(t) = 0 , \\ \dot{p}(t) &= -\nabla c(x)/c(x) , \qquad \dot{M}(t) = -D - MB - B^{\mathsf{T}}M - MCM , \\ \dot{a}_0(t) &= \frac{a_0}{2} \left(-c(x)p \cdot \nabla c(x) - c(x)^3 p \cdot Mp + c(x)^2 \operatorname{Tr}[M] \right) , \end{split}$$

where *B*, *C*, *D* are matrix functions involving x, p and c(x).

- ODEs easy to solve numerically.
- Beams easy to evaluate:

$$v(t,y) = \mathbf{a}_0(t)e^{i\phi(t,y-\mathbf{x}(t))/\varepsilon}, \qquad \phi(t,y) = \phi_0(t) + y \cdot \mathbf{p}(t) + \frac{1}{2}y \cdot \mathbf{M}(t)y.$$

Let us thus require that

$$\Phi_t^2 - c(y)^2 |\nabla \Phi|^2 = O(|y - x(t)|^3),$$

 $A_t + c rac{\nabla \Phi \cdot \nabla A}{|\nabla \Phi|} + rac{c^2 \Delta \Phi - \Phi_{tt}}{2c |\nabla \Phi|} A = O(|y - x(t)|),$

 \Rightarrow We obtain ODEs for ϕ_0 , x, p, M, a_0 .

$$\begin{split} \dot{x}(t) &= c(x)^2 p , \qquad \dot{\phi}_0(t) = 0 , \\ \dot{p}(t) &= -\nabla c(x)/c(x) , \qquad \dot{M}(t) = -D - MB - B^{\mathsf{T}}M - MCM , \\ \dot{a}_0(t) &= \frac{a_0}{2} \left(-c(x)p \cdot \nabla c(x) - c(x)^3 p \cdot Mp + c(x)^2 \mathrm{Tr}[M] \right) , \end{split}$$

 \Rightarrow Asymptotic order of accuracy is

$$v_{tt} - c(y)^2 \Delta v = O\left(\frac{1}{\sqrt{\varepsilon}}\right)$$

Olof Runborg (KTH)
More generally, we can construct higher order beams. Let

$$\mathbf{v}(t,\mathbf{y}) = \mathbf{a}(t,\mathbf{y}-\mathbf{x}(t))\mathbf{e}^{i\phi(t,\mathbf{y}-\mathbf{x}(t))/\varepsilon},$$

where, for order *K* beams,

More generally, we can construct higher order beams. Let

$$\mathbf{v}(t,\mathbf{y}) = \mathbf{a}(t,\mathbf{y}-\mathbf{x}(t))\mathbf{e}^{i\phi(t,\mathbf{y}-\mathbf{x}(t))/\varepsilon},$$

where, for order K beams,

• The phase is a Taylor polynomial of order K + 1,

$$\phi(t,y) = \phi_0(t) + y \cdot p(t) + y \cdot \frac{1}{2}M(t)y + \sum_{|\beta|=3}^{K+1} \frac{1}{\beta!}\phi_\beta(t)y^\beta.$$

(I) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1))

More generally, we can construct higher order beams. Let

$$\mathbf{v}(t,\mathbf{y}) = \mathbf{a}(t,\mathbf{y}-\mathbf{x}(t))\mathbf{e}^{i\phi(t,\mathbf{y}-\mathbf{x}(t))/\varepsilon},$$

where, for order *K* beams,

• The phase is a Taylor polynomial of order K + 1,

$$\phi(t,y) = \phi_0(t) + y \cdot p(t) + y \cdot \frac{1}{2}M(t)y + \sum_{|\beta|=3}^{K+1} \frac{1}{\beta!}\phi_\beta(t)y^\beta.$$

• A is now a finite WKB expansion,

$$a(t,y) = \sum_{j=0}^{\lceil K/2 \rceil - 1} \varepsilon^j a_j(t,y)$$

More generally, we can construct higher order beams. Let

$$\mathbf{v}(t,\mathbf{y}) = \mathbf{a}(t,\mathbf{y}-\mathbf{x}(t))\mathbf{e}^{i\phi(t,\mathbf{y}-\mathbf{x}(t))/\varepsilon},$$

where, for order K beams,

• The phase is a Taylor polynomial of order K + 1,

$$\phi(t,y) = \phi_0(t) + y \cdot p(t) + y \cdot \frac{1}{2}M(t)y + \sum_{|\beta|=3}^{K+1} \frac{1}{\beta!}\phi_\beta(t)y^\beta.$$

• A is now a finite WKB expansion,

$$a(t, y) = \sum_{j=0}^{\lceil K/2 \rceil - 1} \varepsilon^j a_j(t, y)$$

• Each amplitude term a_j is a Taylor polynomial to order K - 2j - 1

$$a_j(t, y) = \sum_{|\beta|=0}^{K-2j-1} \frac{1}{\beta!} a_{j,\beta}(t) y^{\beta}$$

We now require that

- $\Phi(t, y) = \phi(t, y x)$ solves eikonal equation to order $|y x|^{K+2}$
- $a_j(t, y x)$ solve higher order transport equations to order $|y x|^{K-2j}$

< 口 > < 同 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ >

We now require that

- $\Phi(t, y) = \phi(t, y x)$ solves eikonal equation to order $|y x|^{K+2}$
- $a_j(t, y x)$ solve higher order transport equations to order $|y x|^{K-2j}$

Again, this gives ODEs for all Taylor coefficients,

$$\begin{split} \dot{x}(t) &= c(x)^2 p , \qquad \dot{\phi}_0(t) = 0 , \\ \dot{p}(t) &= -\nabla c(x)/c(x) , \qquad \dot{M}(t) = -D - MB - B^{\mathsf{T}}M - MCM , \\ \dot{a}_{j,\beta}(t) &= \dots, \qquad \dot{\phi}_\beta(t) = \dots, \end{split}$$

< 口 > < 同 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ >

We now require that

- $\Phi(t, y) = \phi(t, y x)$ solves eikonal equation to order $|y x|^{K+2}$
- $a_j(t, y x)$ solve higher order transport equations to order $|y x|^{K-2j}$

Again, this gives ODEs for all Taylor coefficients,

$$\begin{split} \dot{x}(t) &= \boldsymbol{c}(x)^2 \boldsymbol{p} , & \dot{\phi}_0(t) = 0 , \\ \dot{\boldsymbol{p}}(t) &= -\nabla \boldsymbol{c}(x)/\boldsymbol{c}(x) , & \dot{\boldsymbol{M}}(t) = -\boldsymbol{D} - \boldsymbol{M}\boldsymbol{B} - \boldsymbol{B}^\mathsf{T}\boldsymbol{M} - \boldsymbol{M}\boldsymbol{C}\boldsymbol{M} , \\ \dot{\boldsymbol{a}}_{j,\beta}(t) &= \dots, & \dot{\phi}_\beta(t) = \dots, \end{split}$$

Asymptotic order of accuracy is

$$v_{tt} - c(y)^2 \Delta v = O\left(\varepsilon^{K/2-1}\right)$$

Properties



 $\mathbf{v}(t,\mathbf{y}) = \mathbf{a}_0(t)\mathbf{e}^{i\phi(t,\mathbf{y}-\mathbf{x}(t))/\varepsilon}, \qquad \phi(t,\mathbf{y}) = \phi_0(t) + \mathbf{y} \cdot \mathbf{p}(t) + \frac{1}{2}\mathbf{y} \cdot \mathbf{M}(t)\mathbf{y}$

Olof Runborg (KTH)

Gaussian Beam Approximation

IPP Garching, 2013 14 / 36

э

イロト イヨト イヨト イヨト

Properties



- $v(t,y) = a_0(t)e^{i\phi(t,y-x(t))/\varepsilon}, \qquad \phi(t,y) = \phi_0(t) + y \cdot \rho(t) + \frac{1}{2}y \cdot M(t)y$
 - $\Phi(t, x(t)) = \phi(t, 0) = \phi_0(t)$ is real valued

< 口 > < 同 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ >

Properties



$$\mathbf{v}(t,\mathbf{y}) = \mathbf{a}_0(t)\mathbf{e}^{i\phi(t,\mathbf{y}-\mathbf{x}(t))/\varepsilon}, \qquad \phi(t,\mathbf{y}) = \phi_0(t) + \mathbf{y} \cdot \mathbf{p}(t) + \frac{1}{2}\mathbf{y} \cdot \mathbf{M}(t)\mathbf{y}$$

- $\Phi(t, \mathbf{x}(t)) = \phi(t, 0) = \phi_0(t)$ is real valued
- If *M*(0) is symmetric and ℑ*M*(0) is positive definite then this is true for *M*(*t*) (which exists) for all *t* > 0.

Properties



$$\mathbf{v}(t,\mathbf{y}) = \mathbf{a}_0(t)\mathbf{e}^{i\phi(t,\mathbf{y}-\mathbf{x}(t))/\varepsilon}, \qquad \phi(t,\mathbf{y}) = \phi_0(t) + \mathbf{y} \cdot \mathbf{p}(t) + \frac{1}{2}\mathbf{y} \cdot \mathbf{M}(t)\mathbf{y}$$

- $\Phi(t, \mathbf{x}(t)) = \phi(t, 0) = \phi_0(t)$ is real valued
- If *M*(0) is symmetric and ℑ*M*(0) is positive definite then this is true for *M*(*t*) (which exists) for all *t* > 0.
- $a_0(t)$ exists everywhere (no blow-up at caustics)

Properties



$$\mathbf{v}(t,\mathbf{y}) = \mathbf{a}_0(t)\mathbf{e}^{i\phi(t,\mathbf{y}-\mathbf{x}(t))/\varepsilon}, \qquad \phi(t,\mathbf{y}) = \phi_0(t) + \mathbf{y} \cdot \mathbf{p}(t) + \frac{1}{2}\mathbf{y} \cdot \mathbf{M}(t)\mathbf{y}$$

- $\Phi(t, \mathbf{x}(t)) = \phi(t, 0) = \phi_0(t)$ is real valued
- If *M*(0) is symmetric and ℑ*M*(0) is positive definite then this is true for *M*(*t*) (which exists) for all *t* > 0.
- $a_0(t)$ exists everywhere (no blow-up at caustics)
- Shape of beam remains Gaussian

Properties



$$\mathbf{v}(t, \mathbf{y}) = \mathbf{a}_0(t)\mathbf{e}^{i\phi(t, \mathbf{y} - \mathbf{x}(t))/\varepsilon}, \qquad \phi(t, \mathbf{y}) = \phi_0(t) + \mathbf{y} \cdot \mathbf{p}(t) + \frac{1}{2}\mathbf{y} \cdot \mathbf{M}(t)\mathbf{y}$$

- $\Phi(t, x(t)) = \phi(t, 0) = \phi_0(t)$ is real valued
- If *M*(0) is symmetric and ℑ*M*(0) is positive definite then this is true for *M*(*t*) (which exists) for all *t* > 0.
- *a*₀(*t*) exists everywhere (no blow-up at caustics)
- Shape of beam remains Gaussian
- For high order need cutoff in a neighborhood of central ray to avoid spurious growth.

$$\mathbf{v}(t,\mathbf{y}) = \mathbf{a}(t,\mathbf{y}-\mathbf{x}(t))\mathbf{e}^{i\phi(t,\mathbf{y}-\mathbf{x}(t))/\varepsilon}\varrho(\mathbf{y}-\mathbf{x}(t))$$

To approximate more general solutions, use superpositions of beams. Let v(t, y; z) be a beam starting from the point y = z and define

$$u_{GB}(t, y) = \varepsilon^{-\frac{n}{2}} \int_{K_0} v(t, y; z) dz$$

 $(n - \text{dimension}, K_0 - \text{compact set})$

To approximate more general solutions, use superpositions of beams. Let v(t, y; z) be a beam starting from the point y = z and define

$$u_{GB}(t,y) = \varepsilon^{-\frac{n}{2}} \int_{\mathcal{K}_0} a_0(t;z) e^{\frac{i}{\varepsilon} \left[\phi_0(t;z) + (y - x(t;z)) \cdot \rho(t;z) + \frac{1}{2}(y - x(t;z) \cdot M(t;z)(y - x(t;z)))\right]}$$

 $(n - \text{dimension}, K_0 - \text{compact set})$

To approximate more general solutions, use superpositions of beams. Let v(t, y; z) be a beam starting from the point y = z and define

$$u_{GB}(t,y) = \varepsilon^{-\frac{n}{2}} \int_{\mathcal{K}_0} a_0(t;z) e^{\frac{i}{\varepsilon} [\phi_0(t;z) + (y - x(t;z)) \cdot \rho(t;z) + \frac{1}{2}(y - x(t;z) \cdot M(t;z)(y - x(t;z)))}$$

 $(n - \text{dimension}, K_0 - \text{compact set})$

• By linearity of the wave equation equation a sum of solutions is also a solution.

To approximate more general solutions, use superpositions of beams. Let v(t, y; z) be a beam starting from the point y = z and define

$$u_{GB}(t,y) = \varepsilon^{-\frac{n}{2}} \int_{\mathcal{K}_0} a_0(t;z) e^{\frac{i}{\varepsilon} \left[\phi_0(t;z) + (y - x(t;z)) \cdot \rho(t;z) + \frac{1}{2}(y - x(t;z) \cdot M(t;z)(y - x(t;z)))\right]}$$

 $(n - \text{dimension}, K_0 - \text{compact set})$

- By linearity of the wave equation equation a sum of solutions is also a solution.
- *u*_{GB}(*t*, *y*) is an asymptotic solution with initial data *u*_{GB}(0, *y*).
 Sufficient to describe e.g. WKB data: ∃ *K*-th order beams s.t.

$$\left\| A(\mathbf{y}) e^{i\phi(\mathbf{y})/\varepsilon} - u_{GB}(\mathbf{0}, \cdot) \right\|_{E} = O(\varepsilon^{K/2}),$$

[Tanushev, 2007].

To approximate more general solutions, use superpositions of beams. Let v(t, y; z) be a beam starting from the point y = z and define

$$u_{GB}(t,y) = \varepsilon^{-\frac{n}{2}} \int_{\mathcal{K}_0} a_0(t;z) e^{\frac{i}{\varepsilon} \left[\phi_0(t;z) + (y - x(t;z)) \cdot \rho(t;z) + \frac{1}{2}(y - x(t;z) \cdot M(t;z)(y - x(t;z)))\right]}$$

 $(n - \text{dimension}, K_0 - \text{compact set})$

- By linearity of the wave equation equation a sum of solutions is also a solution.
- *u*_{GB}(*t*, *y*) is an asymptotic solution with initial data *u*_{GB}(0, *y*).
 Sufficient to describe e.g. WKB data: ∃ *K*-th order beams s.t.

$$\left\| A(y) e^{i\phi(y)/\varepsilon} - u_{GB}(0, \cdot) \right\|_{E} = O(\varepsilon^{K/2}),$$

[Tanushev, 2007].

• Prefactor normalizes beams appropriately, $||u_{GB}||_E = O(1)$.

More general phase space superposition:

Let v(t, y; z, p) be a beam starting from the point y = z with momentum p and define

$$u_{GB}(t,y) = \varepsilon^{-n} \int_{\tilde{K}_0} v(t,y;z,p) dz dp.$$

< 口 > < 同 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ >

More general phase space superposition:

Let v(t, y; z, p) be a beam starting from the point y = z with momentum p and define

$$u_{GB}(t,y) = \varepsilon^{-n} \int_{\tilde{K}_0} v(t,y;z,p) dz dp.$$

 $u_{GB}(t, y)$ is an asymptotic solution with initial data

$$u_{GB}(0,y) = \varepsilon^{-n} \int_{\tilde{K}_0} v(0,y;z,p) dz dp.$$

Can describe more general data. (C.f. FBI transform.)

< 口 > < 同 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ >

Numerical methods

• Approximate superposition integral by sum (trapezoidal rule)

$$u_{GB}(t,y) = \varepsilon^{-\frac{n}{2}} \int_{K_0} v(t,y;z) dz \approx \varepsilon^{-\frac{n}{2}} \sum_j v(t,y;z_j) \Delta z^n.$$

(I) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1))

Approximate superposition integral by sum (trapezoidal rule)

$$u_{GB}(t,y) = \varepsilon^{-\frac{n}{2}} \int_{K_0} v(t,y;z) dz \approx \varepsilon^{-\frac{n}{2}} \sum_j v(t,y;z_j) \Delta z^n.$$

 Lagrangian methods – Solve ODEs ∀z_j with standard methods. Similar to ray tracing but with all the additional Taylor coefficients computed along the rays (*M*, *a*_{j,β}, φ_β, ...) [Hill, Klimes, ...]

ㅋㅋ イヨト

Approximate superposition integral by sum (trapezoidal rule)

$$u_{GB}(t,y) = \varepsilon^{-\frac{n}{2}} \int_{K_0} v(t,y;z) dz \approx \varepsilon^{-\frac{n}{2}} \sum_j v(t,y;z_j) \Delta z^n.$$

- Lagrangian methods Solve ODEs ∀z_j with standard methods. Similar to ray tracing but with all the additional Taylor coefficients computed along the rays (*M*, *a*_{j,β}, φ_β, ...) [Hill, Klimes, ...]
- Eulerian methods obtain parameters from solving PDEs on fixed grids [Leung, Qian, Burridge,07]], [Jin, Wu, Yang,08], [Jin, Wu, Yang, Huang, 09], [Leung, Qian,09], [Qian,Ying,10],...

• Approximate superposition integral by sum (trapezoidal rule)

$$u_{GB}(t,y) = \varepsilon^{-\frac{n}{2}} \int_{\mathcal{K}_0} v(t,y;z) dz \approx \varepsilon^{-\frac{n}{2}} \sum_j v(t,y;z_j) \Delta z^n.$$

- Lagrangian methods Solve ODEs ∀z_j with standard methods. Similar to ray tracing but with all the additional Taylor coefficients computed along the rays (*M*, a_{j,β}, φ_β, ...) [Hill, Klimes, ...]
- Eulerian methods obtain parameters from solving PDEs on fixed grids [Leung, Qian, Burridge,07]], [Jin, Wu, Yang,08], [Jin, Wu, Yang, Huang, 09], [Leung, Qian,09], [Qian,Ying,10],...
- Wavefront methods solve for parameters on a wave front [Motamed, OR,09]

$$u_{GB}(t,y) = \varepsilon^{-\frac{n}{2}} \int_{\mathcal{K}_0} v(t,y;z) dz \approx \varepsilon^{-\frac{n}{2}} \sum_j v(t,y;z_j) \Delta z^n.$$

Cost ~ number of beams since each beam is O(1).
 For accuracy need Δz ~ √ε ~ width of beams.
 ⇒ cost ~ O(ε^{-n/2})
 C.f. direct solution of wave equations, at least O(ε⁻⁽ⁿ⁺¹⁾)

$$u_{GB}(t,y) = \varepsilon^{-\frac{n}{2}} \int_{\mathcal{K}_0} v(t,y;z) dz \approx \varepsilon^{-\frac{n}{2}} \sum_j v(t,y;z_j) \Delta z^n.$$

- Cost ~ number of beams since each beam is O(1). For accuracy need Δz ~ √ε ~ width of beams.
 ⇒ cost ~ O(ε^{-n/2})
 C.f. direct solution of wave equations, at least O(ε⁻⁽ⁿ⁺¹⁾)
- For phase space superposition would get ~ O(ε⁻ⁿ) but can often be improved (e.g. support in p ~ √ε for WKB data)

ㅋㅋ イヨト

$$u_{GB}(t,y) = \varepsilon^{-\frac{n}{2}} \int_{\mathcal{K}_0} v(t,y;z) dz \approx \varepsilon^{-\frac{n}{2}} \sum_j v(t,y;z_j) \Delta z^n.$$

- Cost ~ number of beams since each beam is O(1). For accuracy need Δz ~ √ε ~ width of beams.
 ⇒ cost ~ O(ε^{-n/2})
 C.f. direct solution of wave equations, at least O(ε⁻⁽ⁿ⁺¹⁾)
- For phase space superposition would get ~ O(ε⁻ⁿ) but can often be improved (e.g. support in p ~ √ε for WKB data)
- Spreading of beams
 Wide beams ⇒ large Taylor approximation errors

ヨトイヨト

$$u_{GB}(t,y) = \varepsilon^{-\frac{n}{2}} \int_{K_0} v(t,y;z) dz \approx \varepsilon^{-\frac{n}{2}} \sum_j v(t,y;z_j) \Delta z^n.$$

- Cost ~ number of beams since each beam is O(1).
 For accuracy need Δz ~ √ε ~ width of beams.
 ⇒ cost ~ O(ε^{-n/2})
 C.f. direct solution of wave equations, at least O(ε⁻⁽ⁿ⁺¹⁾)
- For phase space superposition would get ~ O(ε⁻ⁿ) but can often be improved (e.g. support in p ~ √ε for WKB data)
- Spreading of beams
 Wide beams ⇒ large Taylor approximation errors
- Initial data approximation Many degrees of freedom. Can have huge impact on accuracy at later times.

Approximation errors

Let

$$\Box := \partial_{tt} + c(y)^2 \Delta.$$

Suppose u is exact solution of wave equation and \tilde{u} is the Gaussian beam approximation

$$\Box u = 0, \qquad \Box \tilde{u} = O(\varepsilon^{K/2-1}).$$

What is the norm error in \tilde{u} , i.e. $||u - \tilde{u}||$?

(I) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1))

Approximation errors

Let

$$\Box := \partial_{tt} + c(y)^2 \Delta.$$

Suppose u is exact solution of wave equation and \tilde{u} is the Gaussian beam approximation

$$\Box u = 0, \qquad \Box \tilde{u} = O(\varepsilon^{K/2-1}).$$

What is the norm error in \tilde{u} , i.e. $||u - \tilde{u}||$?

Use ε -scaled energy norm

$$||u||_{E}^{2} := \frac{\varepsilon^{2}}{2} \int_{\mathbb{R}^{n}} |u_{t}|^{2} c(y)^{-2} + |\nabla u|^{2} dy.$$

This is O(1) for WKB type initial data,

$$u(0,x) = A(x)e^{i\phi(x)/\varepsilon} \quad \Rightarrow \quad ||u(0,\cdot)||_E = O(1).$$

Use well-posedness (stability) estimate for wave equation solutions w:

$$\|w(t,\cdot)\|_E \leq \|w(0,\cdot)\|_E + \varepsilon C(T) \sup_{t\in[0,T]} \|\Box w(t,\cdot)\|_{L^2}, \qquad 0 \leq t \leq T.$$

Use well-posedness (stability) estimate for wave equation solutions w:

$$\|w(t,\cdot)\|_E \leq \|w(0,\cdot)\|_E + \varepsilon C(T) \sup_{t\in[0,T]} \|\Box w(t,\cdot)\|_{L^2}, \qquad 0 \leq t \leq T.$$

Since, $\Box u = 0$ and by linearity

$$\Box[\tilde{u}-u]=\Box\tilde{u}.$$

Hence, assuming $u(0, x) = \tilde{u}(0, x)$,

$$||\tilde{u}(t,\cdot)-u(t,\cdot)||_{\mathcal{E}} \leq \varepsilon C(T) \sup_{t\in[0,T]} ||\Box \tilde{u}(t,\cdot)||_{L^2}, \quad 0 \leq t \leq T.$$

Error in \tilde{u} ~ how well it satisfies equation, plus one order in ε

Olof Runborg (KTH)

글 🕨 🖌 글

A D M A A A M M

Approximation Errors Single Gaussian Beam

By earlier construction

$$\Box \tilde{u}_{GB}(t,x) = O(\varepsilon^{K/2-1}).$$

and

$$||\tilde{u}(t,\cdot)-u(t,\cdot)||_{E} \leq \varepsilon C(T) \sup_{t\in[0,T]} ||\Box \tilde{u}(t,\cdot)||_{L^{2}}, \qquad 0 \leq t \leq T.$$

(I) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1)) < ((1))

Approximation Errors Single Gaussian Beam

By earlier construction

$$\Box \tilde{u}_{GB}(t,x) = O(\varepsilon^{K/2-1}).$$

and

$$||\tilde{u}(t,\cdot)-u(t,\cdot)||_{E} \leq \varepsilon C(T) \sup_{t\in[0,T]} ||\Box \tilde{u}(t,\cdot)||_{L^{2}}, \qquad 0 \leq t \leq T.$$

Hence, after appropriate normalization (so $||\tilde{u}||_{E} = 1$) and assuming zero initial data error, [Ralston, 82]

$$\sup_{t\in[0,T]} ||\tilde{u}(t,\cdot)-u(t,\cdot)||_{\mathsf{E}} \leq O(\varepsilon^{K/2}).$$

Estimate is sharp.

Approximation Errors Superpositions

Norm estimates of $||u - u_{GB}||$ only rather recently derived [Swart, Rousse, Liu, Ralston, Tanushev, Bougacha, Alexandre, Lu, Yang,...]

Need to check how well u_{GB} satisfies equation

 $\Box u_{GB} := \partial_{tt} u_{GB} + c(y)^2 \Delta u_{GB}, \qquad u_{GB}(t,y) = \varepsilon^{-\frac{n}{2}} \int_{K_0} v(t,y;z) dz.$

Approximation Errors Superpositions

Norm estimates of $||u - u_{GB}||$ only rather recently derived [Swart, Rousse, Liu, Ralston, Tanushev, Bougacha, Alexandre, Lu, Yang,...]

Need to check how well u_{GB} satisfies equation

 $\Box u_{GB} := \partial_{tt} u_{GB} + c(y)^2 \Delta u_{GB}, \qquad u_{GB}(t,y) = \varepsilon^{-\frac{n}{2}} \int_{\mathcal{K}_0} v(t,y;z) dz.$

• By linearity,

$$\Box u_{GB} = \varepsilon^{-n/2} \int_{K_0} \Box v(t, y; z) dz.$$
Approximation Errors Superpositions

Norm estimates of $||u - u_{GB}||$ only rather recently derived [Swart, Rousse, Liu, Ralston, Tanushev, Bougacha, Alexandre, Lu, Yang,...]

Need to check how well u_{GB} satisfies equation

 $\Box u_{GB} := \partial_{tt} u_{GB} + c(y)^2 \Delta u_{GB}, \qquad u_{GB}(t,y) = \varepsilon^{-\frac{n}{2}} \int_{K_0} v(t,y;z) dz.$

• By linearity and simple estimate,

$$|\Box u_{GB}| \leq \varepsilon^{-n/2} \int_{\mathcal{K}_0} |\Box v(t, y; z)| dz.$$

Approximation Errors Superpositions

Norm estimates of $||u - u_{GB}||$ only rather recently derived [Swart, Rousse, Liu, Ralston, Tanushev, Bougacha, Alexandre, Lu, Yang,...]

Need to check how well u_{GB} satisfies equation

 $\Box u_{GB} := \partial_{tt} u_{GB} + c(y)^2 \Delta u_{GB}, \qquad u_{GB}(t,y) = \varepsilon^{-\frac{n}{2}} \int_{K_0} v(t,y;z) dz.$

• By linearity and simple estimate,

$$|\Box u_{GB}| \leq \varepsilon^{-n/2} \int_{K_0} |\Box v(t,y;z)| dz.$$

• After scaling and Cauchy–Schwarz,

$$||\Box u_{GB}(y)||_2^2 \leq C \varepsilon^{-n} \int_{\mathcal{K}_0} ||\Box v(t,y;z)||_2^2 dz \leq C \varepsilon^{-n} \varepsilon^{\mathcal{K}-2+n/2}$$

Approximation Errors Superpositions

Norm estimates of $||u - u_{GB}||$ only rather recently derived [Swart, Rousse, Liu, Ralston, Tanushev, Bougacha, Alexandre, Lu, Yang,...]

Need to check how well u_{GB} satisfies equation

 $\Box u_{GB} := \partial_{tt} u_{GB} + c(y)^2 \Delta u_{GB}, \qquad u_{GB}(t,y) = \varepsilon^{-\frac{n}{2}} \int_{K_0} v(t,y;z) dz.$

• By linearity and simple estimate,

$$|\Box u_{GB}| \leq \varepsilon^{-n/2} \int_{K_0} |\Box v(t, y; z)| dz.$$

• After scaling and Cauchy–Schwarz,

$$||\Box u_{GB}(y)||_2^2 \leq C \varepsilon^{-n} \int_{\mathcal{K}_0} ||\Box v(t,y;z)||_2^2 dz \leq C \varepsilon^{-n} \varepsilon^{\mathcal{K}-2+n/2}$$

Gives error estimate for u_{GB}

$$||u - u_{GB}||_{E} \leq \varepsilon C ||\Box u_{GB}||_{2} \leq C \varepsilon^{K/2 - n/4}.$$

イロト 不得 トイヨト イヨト ニヨー

Basic estimate

$$||u(t,\cdot)-u_{GB}(t,\cdot)||_{E} \leq O(\varepsilon^{K/2-n/4})$$
.

is not sharp. E.g. it does not predict convergence for first order beams in 2D.

Basic estimate

$$||u(t,\cdot) - u_{GB}(t,\cdot)||_E \leq O(\varepsilon^{K/2-n/4}).$$

is not sharp. E.g. it does not predict convergence for first order beams in 2D.

• Problem: The step

$$|\Box u_{GB}| \leq \varepsilon^{-n/2} \int_{K_0} |\Box v(t, y; z)| dz$$

ignores cancellations between neighbouring beams. Very bad except at caustics where beams interfere constructively.

글 🕨 🖌 글

Basic estimate

$$||u(t,\cdot) - u_{GB}(t,\cdot)||_E \leq O(\varepsilon^{K/2-n/4}).$$

is not sharp. E.g. it does not predict convergence for first order beams in 2D.

• Problem: The step

$$|\Box u_{GB}| \leq \varepsilon^{-n/2} \int_{\mathcal{K}_0} |\Box v(t, y; z)| dz$$

ignores cancellations between neighbouring beams. Very bad except at caustics where beams interfere constructively.

• Gives a dependence on dimension *n* in estimate.

For the wave equation,

$$||u(t,\cdot) - u_{GB}(t,\cdot)||_E \leq O(\varepsilon^{K/2})$$
.

For the Schrödinger equation,

 $||u(t,\cdot)-u_{GB}(t,\cdot)||_{L^2} \leq O(\varepsilon^{K/2})$.

3

For the wave equation,

$$||u(t,\cdot) - u_{GB}(t,\cdot)||_E \leq O(\varepsilon^{K/2})$$
.

For the Schrödinger equation,

```
||u(t,\cdot)-u_{GB}(t,\cdot)||_{L^2} \leq O(\varepsilon^{K/2}).
```

 Superposition in physical space. Initial data approximated on a submanifold of phase space (WKB data).

For the wave equation,

$$||u(t,\cdot) - u_{GB}(t,\cdot)||_E \leq O(\varepsilon^{K/2})$$
.

For the Schrödinger equation,

```
||u(t,\cdot)-u_{GB}(t,\cdot)||_{L^2} \leq O(\varepsilon^{K/2}).
```

- Superposition in physical space. Initial data approximated on a submanifold of phase space (WKB data).
- Convergence of all beams independent of dimension and presence of caustics.

For the wave equation,

$$||u(t,\cdot) - u_{GB}(t,\cdot)||_E \leq O(\varepsilon^{K/2})$$
.

For the Schrödinger equation,

```
||u(t,\cdot)-u_{GB}(t,\cdot)||_{L^2} \leq O(\varepsilon^{K/2}).
```

- Superposition in physical space. Initial data approximated on a submanifold of phase space (WKB data).
- Convergence of all beams independent of dimension and presence of caustics.
- Result also for general scalar, strictly hyperbolic *m*-th order PDEs.

For the wave equation,

$$||u(t,\cdot) - u_{GB}(t,\cdot)||_E \leq O(\varepsilon^{K/2})$$
.

For the Schrödinger equation,

```
||u(t,\cdot)-u_{GB}(t,\cdot)||_{L^2} \leq O(\varepsilon^{K/2}).
```

- Superposition in physical space. Initial data approximated on a submanifold of phase space (WKB data).
- Convergence of all beams independent of dimension and presence of caustics.
- Result also for general scalar, strictly hyperbolic *m*-th order PDEs.
- Cf. [Bougacha, Akian, Alexandre, 2009], [Rousse, Swart, 2009] and [Lu, Yang, 2011] for other settings.

For time-harmonic waves consider Helmholtz equation

$$\Delta u + (i\alpha \varepsilon^{-1} + \varepsilon^{-2})n^2 u = g, \qquad x \in \mathbb{R}^d.$$

where n(x) = 1/c(x), α =damping and g supported on a co-dimension one manifold. (Ex. $g = g_0(x_2)\delta(x_1)/\varepsilon$.)

イロト イ押ト イヨト イヨト

For time-harmonic waves consider Helmholtz equation

$$\Delta u + (i\alpha \varepsilon^{-1} + \varepsilon^{-2})n^2 u = g, \qquad x \in \mathbb{R}^d.$$

where n(x) = 1/c(x), α =damping and g supported on a co-dimension one manifold. (Ex. $g = g_0(x_2)\delta(x_1)/\varepsilon$.)



the second se																	
a contract the second second																	
and the second sec																	
the second se																	
the second se																	
the second se																	
the second se																	
the second se																	
the second se																	
the second se																	
the second se																	
the second the second the second the second terms and the second terms are set of the second terms and terms are set of the second terms are set of terms are set o																	
the second se																	
the second se																	
the second se																	
a state the second state of the second state o																	
and the second se															_		
the second se												-			-		
												_			_		
the second se			_						_			_					
the second se	_		_		_	_	_		_			_			-		
the second second	_	_	_		_	_	_		_	_		_	_	_	_		
the second se	_	_				_	_			_			_				
	_				-				-			-			-		
						_	_			_							
	_	_	-		_	_	_		-			-		_	-		-
	_		-						-			-			-		-
					_	_	_										
	_	_	-		_	_			-			-		_	-		_
	_		-		_				-			-		-	_		-
			_		_				_			_			_		
	_		_						_								
				_													
	_		-		-				-		-	-		-	-		-
		-	-	-	-	: =			-		ε.	=	6.3	€.	-	63	=
			3			1.18		ε.	=		ε.	3	6.3	€.	3	Ε3	s
			르		. 3	13	13	ε.	물	13	ε.	3	13	£.	3	E	율
	18		3	1	3	13		E :	3	13	E.	3	13	Ē	3	H	ŝ
	18		3	1	3	18		E	3	1	E.	3		Ē	3	H	8
	1		3	3		13		Ŀ	1		ŧ.	3		ŧ.		H	0000
		8	NNN	1				L	WHM	1	E	NNN			1	U	WHH
		1	WHR		and a		1	L	WHH	in the second	E.	NNNN		and a	NNN	U	WHHH
	and a	1	WHEN		in the second		in the second	L	WHHH	in the second	E.	NNNN		WHAT	NNNN	U	WHHH
		1	NAME:	line.	innin in	1 milli	in the second	L	WHIN	in in	L	HHHH		WHAT	HANNA	U	TERMINA
	No.	1	MARKET	ill.	innin i	iline a	in the second	L	MARKED	in in		NAMES	iiiiiiiiiiiiiiiiiiiiiiiiiiiiiiiiiiiiii	WHIMMAN	WHEN		WHIMMAN
	WARTE	1	MARRIEL !!	littee	in the second	litim	in the second		MMMMM111	Thisises		HARRING	initian in the second s	WHITEL	UNKNESS		TTTTT NUMBER
	No.	1	NAMES OF TAXABLE	lillium	(()))	littem	(itanu		MARRIEL	((()))))		WARRAN	(iiiiiiii)	HINKSON	Unanni		TERFORMENT
111	NHA11		MARRIEL	(()))	(()))))))))	littem	(())))))))))))))))))))))))))))))))))))		MARRIEL	((()))))		NAMES OF COLUMN	This is not a second	MILLION OF	WHERE W		NHARRAN !!!!
	No.		NAMES OF COLUMN	WINNIN	IIIIIIIIII	littem	With Associate		MARRIES !!!	(((())))))))		UNIVERSITY	Numerous and a second	NUMBER OF	WHERE AND		COLUMN AND ADDRESS OF
	Marrie I.	man (ARREST OF THE OWNER OWNE	((()))	((()))))))))))))))))))))))))))))))))))	Illinna	With A and		MARRIEL	Summer of the local division of the local di		NUMBER OF CONTRACTOR	Nillinin	NHAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAA	HIMARKAN		COLUMN AND ADDRESS OF
	No.		NAMES OF COLUMN	((()))	(((()))))))))	(((i)))))	(((())))))))))		MARKED !!!!!	((((()))))))))))))))))))))))))))))))))		NUMBER OF CONTRACTOR	Withhis	NHAMAN	WHERE AND		NAMES OF TAXABLE PARTY.
	MARTIN	mm())))	ARRENT OF THE OWNER	((()))	10000000000000000000000000000000000000	(((i)))))	((((()))))))))))))))))))))))))))))))))		MARKEN	Williams		NUMBER	With Laws	NHARAN L	WHIRE AND A		COLUMN STREET, STORY
	(()))	mm())))	NAMES OF COLUMN	and and a	((((()))))))))))))))))))))))))))))))))	(((((((((((((((((((((((((((((((((((((((((((())))))))))))))))))))))))))))))))		MMMMM111//	((((((((((((((((((((((((((((((((((((((Market Color	VIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIII	With Laws	MARKAGE L	Philippeerses		COLUMN STREET,
	Contraction of the	m	NAMES OF COLUMN	[[[[]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]	T/////////////////////////////////////	(((((())))))))))))))))))))))))))))))))	10000000000000000000000000000000000000		MARRIED //	Thinkson	Manufactor of the local division of the loca	PURISHING STREET	Withhiston	MARGAREN	PHUMBER AND	Number of Street, Stre	TOTAL STREET,
	A CONTRACTOR	million	PUPPERSONAL CONTRACTOR	[[]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]	17110000000000000000000000000000000000	(())))))))))))))))))))))))))))))))))))	17110000000000000000000000000000000000	ANNUAL CONTRACTOR	ANNAL STREET, STREET, ST.	Philipperson	And Addition of the owner of the owner of the owner	F ////////////////////////////////////	WWWWWWWWWWWW	MANAGE STATE	PUMARANA CONTRACTOR	RED LAND LAND	CONTRACTOR OF CONTRACTOR
	MARKED () / / /	mail	PERSONAL PROPERTY OF A DESCRIPTION OF A	Thinking and the second second	17110000000000000000000000000000000000	Contraction of the local distance of the loc	11100000000000000000000000000000000000	A REAL COLORADO	ANNOUNCED (1)	PPRODUCT AND ADDRESS OF	And a state of the local division of the loc	P / / / / / / / / / / / / / / / / / / /	FFFFFFFFFFFFFFFFFFFFFFFFFFFFFFFFFFFFF	MANAGE AND THE PARTY OF	17700000000000000000000000000000000000	CONTRACTOR OF STREET, ST	CONTRACTOR OF A DESCRIPTION OF A DESCRIP
	AND	m	P P P P P P P P P P P P P P P P P P P	The second second	PUTABLE CONTRACTOR	11/1/1/1/1/1/1/1/1/1/1/1/1/1/1/1/1/1/1	T/////////////////////////////////////	ARRITER CONTRACTOR	ANNOUNCES STOLL &	FFFFFFFFFFFFFFFFFFFFFFFFFFFFFFFFFFFFFF	A S S S S S S S S S S S S S S S S S S S	FFFFFFFFFFFFFFFFFFFFFFFFFFFFFFFFFFFFFF	FURNISSIA AND ADDRESS	MANAGE STATE	TTUUMBERSON	REAL COLUMN	MANAGERS AND A TO A T
	AND	anno anno anno anno anno anno anno anno	PERSONAL PROPERTY OF A DESCRIPTION OF A	ALL COMPANY	11/1/10/00/00/00/00/00/00/00/00/00/00/00	Contraction of the second	11110000000000000000000000000000000000	ARRITER OF THE OWNER	ANNOUNCED IN CONTRACTOR OF CON	ITTUNINA AMAN	A STATEMENT OF A DESCRIPTION OF A DESCRI	277710000000000000000000000000000000000	THE RESIDENCE AND ADDRESS OF ADDRES	NAMES AND ADDRESS OF	TTTHINK AND	ARRING CONTRACTOR	NUMBER OF STREET
	MARKED ST. S.	anna anna anna anna anna anna anna ann	ANNAL AND A STREET	ALL CONTRACTOR	11111100000000000000000000000000000000	ATTIC REPART	111111111111111111111111111111	ARRENT COLUMN	NAMES OF TAXABLE PARTY.	(FFFE)	NAMES OF TAXABLE PARTY	A P F F F F F F F F F F F F F F F F F F	ARRENING ALLEND	NAMES AND ADDRESS OF TAXABLE PARTY OF TA	TTTHIUMMENT	HEERING CONTRACTOR	CONTRACTOR OF THE OWNER
	MARKAN CONTRACTOR	And the second se	ANNAL AND A REPORT OF A REPORT	ALL COMPANY	11111100000000000000000000000000000000	APPENDING NUMBER	111111111111111111111111111111	ARRENT COLUMN	NAMES OF COLUMN STREET, ST. C.	(FFF/99900000000000000000000000000000000	AND DESCRIPTION OF	APPENDING CONTRACTOR	ATTRACTICAL CONTRACTOR OF STREET, STRE	AND ADDRESS OF TAXABLE PARTY OF TAXABLE	11111111111111111111111111	ARREN DALLARD	NUMBER OF STREET
	MARKEN (() / / / / / / / / / / / / / / / / /		ANA FEATURE AND A CONTRACTOR OF THE ACCOUNTS	And a second sec	1111111111111111111111111111111	ATT A DESCRIPTION OF A	11111111111111111111111111111111111111	ARRENT COLORADO	NAMES OF A DESCRIPTION OF A DESCRIPTIONO	11111111111111111111111111111111111111	Contraction of the owner owner owner owner owner owner owne	Contraction and a second second	ARRENING ALLEN	NAMES OF TAXABLE PARTY.	11111111111111111111111111111111111111	ARREND ALL	NUMBER OF STREET, STRE
	MARRIEL		NAMES OF A CONTRACT OF A CONTR	And the second se	11111111111111111111111111111111111111	Contraction of the owner own	1111111111111111111111111111111	NAMES OF COLUMN	NAMES OF COLUMN STREET, ST. C.	11/11/11/10/10/10/10/10/10/10/10/10/10/1	NAMES OF COLUMN	TALEFT FOR THE PROPERTY OF THE	INTERVISION AND ADDRESS OF	AND ADDRESS OF TAXABLE	11111111111111111111111111111111111111	WHERE AND A CONTRACT OF A CONT	WWWWWWWWWWWWWWWWWWWWWWWWWWWWW
	NAMES OF TAXABLE PARTY	And the second se	AND A FEATING AND	And a second sec	CONTRACTION NAMES AND ADDRESS OF	A TANK TANK TANK TANK TANK TANK TANK TAN	(1111111000000000000000000000000000000	ANARASIS CONTRACTOR	NAMES OF A POST OFFICE ADDRESS OFFICADOFFICADOFFICADOFFICE ADDRESS OFFICADOFFICAD	11111111111111111111111111111111111111	The second se	COLUMN STRUCTURE STRUCTURE	ULARRENDEREALANCE	AND ADDRESS OF TAXABLE	1111111111111111111111111111111	AMARKANIA AND AND AND AND AND AND AND AND AND AN	CONTRACTOR OF A CONTRACTOR OF
	MARKEN (VIII)		AND A FEATING AND	And a second designment	ALLEY FULLING AND	Contraction of the second	(11/1/1/1/1/1/1/1/1/1/1/1/1/1/1/1/1/1/1	NAMES OF COLUMN STREET, ST. CO.	NAMES OF TAXABLE PARTY.	11111111111111111111111111111111111111	ANNAL COLUMN	COLUMN FOR THE REPORT OF THE PARTY OF THE PA	10.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.0.	AND ADDRESS OF TAXABLE PARTY OF TAXABLE	1111111111111111111111111111111	ANNARANA ANALASI ANA	ANNOUNCE FOR THE REAL OF THE PARTY OF THE PA
	MARKED ST. T. T		NAMES AND ADDRESS OF A DESCRIPTION OF A	Manager Contraction	ALCOST & PRODUCTION AND ADDRESS OF	Contraction of the second	11111111111111111111111111111111111111	NUMBER STOCK	NAMES OF TAXABLE PARTY OF TAXABLE PARTY.	11111111111111111111111111111111111111	NUMBER OF CONTRACTOR OF CONTRA	ALL STATION AND A STATION AND AND A STATION	10100000000000000000000000000000000000	NUMBER OF TAXABLE PARTY OF	11111111111111111111111111111111111111	WWWWWWWWWWWW	NUMBER OF STREET, STRE
	NAMES OF CONTRACTOR OF CONTRAC		NAMES OF A PARTICULAR REPORT	Manager Construction	Contract of the Contract of Co		11111111111111111111111111111111111111	ANNARA CONTRACTOR	NAMES OF TAXABLE PARTY	11111111111111111111111111111111111111	AND A REAL PROPERTY OF A REAL PR	COLORE FERRING REAL FROM	11111111111111111111111111111111111111	NAMES OF TAXABLE PARTY	COLUCT FUTUREMENT	WWWWWWWWWWWWW	NAMES OF TAXABLE PARTY.
		South Contraction of the local division of t	COLOR FFEEDRACH REPORT	Manager Construction	COLUMN FOR CONTRACTOR		11111111111111111111111111111111111111	ANNARA SULLEY STATE	NAMES OF A POST OFFICE A DESCRIPTION OF	11111122111111111111111111111111111111	CONTRACTOR OF A DESCRIPTION OF A DESCRIP	COLORE FEEDRICH REAL FORMATION OF	//////////////////////////////////////	AND ADDRESS OF TAXABLE PARTY OF TAXABLE	COLUMN FERMINISTRAMON	Philippen and a second s	Contract of the Contract of th
	State of Concession, Name	Contraction of the local division of the loc	AND STATES AND STATES AND		COLORA CONTRACTOR		11111111111111111111111111111111111111	PUNNARRAN COLORA	NAMES OF A PROPERTY OF A PROPE	PUTATION CONTRACTOR	ANNALS STATISTICS	COLORE FERMINISTRATION	17/1//////////////////////////////////	AND ADDRESS OF TAXABLE PARTY OF TAXABLE	COLUMN FERRINANA ANALY	PPINAR REPAIRS	AND ADDRESS OF THE R. P. D.
	NAMES OF TAXABLE PARTY	Contraction of the local division of the loc	PARTICIPATION CONTRACTOR		CONTRACTOR CONTRACTOR	Provident and a second second second	44111111111111111111111111111111111111	PRIMARE SUCCESSION OF STREET, STRE	NAMES OF TAXABLE PARTY	######################################	APPROPRIATION OF A PARTY OF A PAR	CONTRACTOR CONTRACTOR	COUNTRANSMISSION	AND ADDRESS OF TAXABLE PARTY OF TAXABLE	FEATURE FERRING AND	FPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPP	NUMBER OF TAXABLE PARTY
	NAMES OF TAXABLE PARTY	State of the local division of the local div			APPLICATION OF APPLICATION AND ADDRESS	A STATE AND DESCRIPTION OF A DESCRIPTION	4 / / / / / / / / / / / / / / /////////	A CONTRACTOR DATE	NAMES OF A PARTY OF A DESCRIPTION OF A D	FULLING FULLING AND ADDRESS AND ADDRESS ADDRES	PERSONAL AND A PERSON AND A PER	CONTRACTOR FURNISHING CONTRACTOR	PUTINTERNICAL AND A CONTRACTOR OF A CONTRACT	AND ADDRESS OF TAXABLE PARTY OF TAXABLE	FULLING FERRINANA CONTRACTOR	PUTTO MARKANIA AND AND AND AND AND AND AND AND AND AN	NAMES OF TAXABLE PARTY
		Contraction of the local division of the loc			A PULLING STATION AND AND AND AND AND AND AND AND AND AN	APPROX AP	4 4 / / / / / / / / / / / / / /////////	APPROPRIATE COLORADO	NAMES OF TAXABLE PARTY OF TAXABLE PARTY.	FFULLINER REPRESENTATION	APPROPRIATE CONTRACTOR OF A	Contraction of the statement of the stat	17/1//////////////////////////////////	AND ADDRESS OF TAXABLE PARTY OF TAXABLE	FELOLOGIEFERING MANAGEMENT	APPOINTER REALIZED AND ADDRESS OF	NAMES OF TAXABLE PARTY OF TAXABLE PARTY.
	NAME OF TAXABLE PARTY O	Contraction of the local division of the loc			A PULLING A PULLING AND	A RECEIPTION OF THE PARTY OF TH	44441111111111111111111111111111111111	A RUNNING COLUMN	APPROX APPROX APPROX APPROX	APPIN NAME APPROXIMATION	A PERSONAL AND A PERSONAL PROPERTY OF A PERSO	A PERSONAL PROPERTY AND A REPORT OF A PERSONAL PROPERTY AND A PERSONAL PROPERTY A PERSONAL PROPERTY AND A PERSONAL PROPERTY AND A PERSONAL PROPERTY A PERSONAL PROPERTY A PERSONAL PROPERTY A PERSONAL PROPERTY AND A PERSONAL PROPERTY A PERSONAL PROPERTY AND A PERSONAL PROPERTY A PERSONAL PROPERTY A PERSONAL PROPERTY A PERSONAL PROPERTY A PERSONAL	APPINTARE RESIDENCES	AND ADDRESS OF TAXABLE PARTY OF TAXABLE	FEELOOFFFFFFFFFFFFFFFFFFFFFFFFFFFFFFFFF	APPONTATION CONTRACTOR	NAMES OF TAXABLE PARTY
		Contraction of the local division of the loc			AND AND AND AND ADDRESS OF ADDRES	And a second sec	11111111111111111111111111111111111111	14441000000000000000000000000000000000	NAMES OF TAXABLE PARTY OF TAXABLE PARTY.	11111111111111111111111111111111111111	A & FUILDER STATISTICS	A & & F & F & F & F & F & F & F & F & F	APPIN APPARTMENT	NAMES AND ADDRESS OF TAXABLE PARTY OF TA	TERVICE TERMINANANANANANANANANANANANANANANANANANANA	A FINNING RUNNING CONTRACTOR	AND ADDRESS OF THE READING TO A DECK
		And the second s			APPROX OF THE PROPERTY OF THE	CAREFORD AND CONTRACTOR OF	APPROX 1011111111111111111111111111111111111	ARRENT CONTRACTOR CONT	NAMES OF TAXABLE PARTY OF TAXABLE PARTY.	CONTRACTOR NUMBER AND ADDRESS OF	APPENDING AND TO THE PARTY OF A DECEMPENDING AND AN	CONTRACTOR OF A PROPERTY OF A PARTY	CARACT/COMPANY/SIGNATION	Contraction of the owner	CONTRACTOR OF PROPERTY OF DESCRIPTION	APPORTANT CONTRACTOR CONTRACTOR	NAMES OF TAXABLE PARTY OF TAXABLE PARTY
				Contraction of the second second	A PERSONNEL PERSONNEL MANAGEMENT	And a second sec	11111111111111111111111111111111111111	ARRENT CONTRACTOR OF A CONTRAC	NAMES OF A CONTRACTOR OF A CONTRACT OF A CON	APPENDATE REPORTS	THEFT PROPERTY AND A PARTY OF A P	TARA CONTRACTOR CONTRACTOR	APPENDATE REPAIRS	CONTRACTOR OF CO	THEFT OF THE PROPERTY OF THE PARTY OF THE PA	NH FF WWWWWWWWWWWWWWW	NAMES OF TAXABLE PARTY OF TAXABLE PARTY.
					ANNUAL CONTRACTOR CONT		11111111111111111111111111111111111111	AAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAA	MANAGE MANUAL PROPERTY AND ADDRESS	11111111111111111111111111111111111111	AN A PRIMARY AND A CONTRACT OF	TARA CONTRACTOR CONTRACTOR	11144711111111111111111111111111111111	Contraction of the second statements	TATE CONTRACTOR OF THE DESIGNATION OF THE DESIGNATI	ANA A PRIVATE REPAIRS	AND REPORTED AND FOUND AND REAL PROPERTY.
					Contraction of the Particulation and the Particulation of the Particulat	CONTRACTOR AND CONTRACTOR	COLUMN TO COLUMN TWO IS NOT THE OWNER.	COMPACTOR DESCRIPTION OF THE OWNER OWNE OWNER	NAMES OF A CONTRACT OF A CONTR	ALTER CONTRACTOR NUMBER AND DE CONTRACTOR	THEFT PROPERTY INCOME.	COLORADO CONTRACTOR CONTRACTOR	ADDREED AND ADDREED AD	Contraction of the owner	CONTRACTOR FERMININAMENT	THEFT WINNERS STATISTICS	CALLER & CONTRACT
					And a second second distances and		11111111111111111111111111111111111111	ANARA PRIMARA PRIMARA PRIMA	NAMES OF A DESCRIPTION OF A DESCRIPTIONO	VIIII I I VIII VIIIIIIIIIIIIIIIIIIIIII	THE REPORT AND A DESCRIPTION OF A DESCRI	THE REPORT OF THE PROPERTY OF	VIIII I VIVIIIIIIIIIIIIIIIIIIIIIIIIIII	AND ADDRESS OF TAXABLE PARTY OF TAXABLE	COLUMN STORY CONTRACTOR STORY	VIIII STANDARDINAL CONTRACTOR	AND A REPORT OF A PARTY AND A PARTY OF A PAR

"Blobs" ⇒ "Fat rays" localized around geometrical optics ray

For time-harmonic waves consider Helmholtz equation

$$\Delta u + (i\alpha \varepsilon^{-1} + \varepsilon^{-2})n^2 u = g, \qquad x \in \mathbb{R}^d.$$

where n(x) = 1/c(x), α =damping and g supported on a co-dimension one manifold. (Ex. $g = g_0(x_2)\delta(x_1)/\varepsilon$.)



• "Blobs" \Rightarrow "Fat rays" localized around geometrical optics ray

To leading order gaussian transversely to ray

Olof Runborg (KTH)

Gaussian Beam Approximation

IPP Garching, 2013 25 / 36

• Same ansatz,

$$v = a(s, y - x(s))e^{i\phi(s, y - x(s))/\varepsilon},$$

centered around a geometrical optics ray x(s) but s not time.

∃ > < ∃</p>

• Same ansatz,

$$v = a(s, y - x(s))e^{i\phi(s, y - x(s))/\varepsilon},$$

centered around a geometrical optics ray x(s) but s not time.

First order beams are of the form

$$\phi = \phi_0(s) + \mathbf{y} \cdot \mathbf{p}(s) + \frac{1}{2}\mathbf{y} \cdot \mathbf{M}(s)\mathbf{y}, \qquad \mathbf{a} = \mathbf{a}_0(s),$$

i.e. a approximated to 0th order, and ϕ to 2nd order.

• Same ansatz,

$$v = a(s, y - x(s))e^{i\phi(s, y - x(s))/\varepsilon},$$

centered around a geometrical optics ray x(s) but s not time.

• First order beams are of the form

$$\phi = \phi_0(s) + \mathbf{y} \cdot \mathbf{p}(s) + \frac{1}{2}\mathbf{y} \cdot \mathbf{M}(s)\mathbf{y}, \qquad \mathbf{a} = \mathbf{a}_0(s),$$

i.e. *a* approximated to 0th order, and ϕ to 2nd order.

• Similar ODEs for a_0, x, p, M, ϕ_0 as in the time-dependent case.

Same ansatz,

$$v = a(s, y - x(s))e^{i\phi(s, y - x(s))/\varepsilon},$$

centered around a geometrical optics ray x(s) but s not time.

• First order beams are of the form

$$\phi = \phi_0(s) + \mathbf{y} \cdot \mathbf{p}(s) + \frac{1}{2}\mathbf{y} \cdot \mathbf{M}(s)\mathbf{y}, \qquad \mathbf{a} = \mathbf{a}_0(s),$$

i.e. *a* approximated to 0th order, and ϕ to 2nd order.

- Similar ODEs for a_0, x, p, M, ϕ_0 as in the time-dependent case.
- Similar properties as in time-dependent case:
 - Phase ϕ evaluated on ray = $\phi_0(s)$ is real valued
 - If *M*(0) is symmetric and ℑ*M*(0) is positive definite then this is true for *M*(*s*) (which exists) for all *s* > 0.
 - $a_0(s)$ exists everywhere (no blow-up at caustics)

Extension off ray

$$v(y) = a(s, y - x(s))e^{i\phi(s, y - x(s))/\varepsilon},$$

• How to evaluate "(s, y - x(s))" in expression for beam?

э

< ロ > < 同 > < 回 > < 回 >

Extension off ray



- How to evaluate "(s, y x(s))" in expression for beam?
- No distinguished "time" variable ⇒ Extend beam by Taylor expansion transversely to ray:

Let $s^* = s^*(y)$ such that $x(s^*)$ is closest point on ray to y.

Extension off ray



- How to evaluate "(s, y x(s))" in expression for beam?
- No distinguished "time" variable ⇒ Extend beam by Taylor expansion transversely to ray:

Let $s^* = s^*(y)$ such that $x(s^*)$ is closest point on ray to y.

Only well-defined close enough to ray ⇒ Cutoff *ρ*(*y*) needed also for first order beams (size η)

< ロ > < 同 > < 回 > < 回 >

Source

Helmholtz with source on $\Sigma = \{y : \rho(y) = 0\}.$

$$\Delta u + (i\alpha\varepsilon^{-1} + \varepsilon^{-2})n^2 u = \frac{1}{\varepsilon}g(y)\delta(\rho(y)).$$



э

Source

Helmholtz with source on $\Sigma = \{y : \rho(y) = 0\}.$

$$\Delta u + (i\alpha\varepsilon^{-1} + \varepsilon^{-2})n^2 u = \frac{1}{\varepsilon}g(y)\delta(\rho(y)).$$



• Beams shoot out orthogonally in each direction from $\boldsymbol{\Sigma}$

Source

Helmholtz with source on $\Sigma = \{y : \rho(y) = 0\}.$

$$\Delta u + (i\alpha\varepsilon^{-1} + \varepsilon^{-2})n^2 u = \frac{1}{\varepsilon}g(y)\delta(\rho(y)).$$



Beams shoot out orthogonally in each direction from Σ
Gives beams v[±](y), with v⁺(y) = 0 when ρ(y) < 0 etc.

Source

Helmholtz with source on $\Sigma = \{y : \rho(y) = 0\}$.

$$Lu =: \Delta u + (i\alpha\varepsilon^{-1} + \varepsilon^{-2})n^2 u = \frac{1}{\varepsilon}g(y)\delta(\rho(y)).$$



Beams shoot out orthogonally in each direction from Σ

- Gives beams $v^{\pm}(y)$, with $v^{+}(y) = 0$ when $\rho(y) < 0$ etc.
- Note that $v^+ = v^-$ on Σ , but $\nabla \phi^+ = -\nabla \phi^-$ so that $L(v^+ + v^-) \sim \delta(\rho(y))$ + smooth part.

Superposition

$$Lu =: \Delta u + (i\alpha\varepsilon^{-1} + \varepsilon^{-2})n^2 u$$
$$= \frac{1}{\varepsilon}g(y)\delta(\rho(y)).$$



Let v[±](y; z) be the beams starting from z ∈ Σ and define superposition

$$u_{GB}(y) = \varepsilon^{-\frac{n-1}{2}} \int_{\Sigma} [v^+(y;z) + v^-(y;z)] dA_z$$
(1)

Superposition

$$Lu =: \Delta u + (i\alpha\varepsilon^{-1} + \varepsilon^{-2})n^2 u$$
$$= \frac{1}{\varepsilon}g(y)\delta(\rho(y)).$$



Let v[±](y; z) be the beams starting from z ∈ Σ and define superposition

$$u_{GB}(y) = \varepsilon^{-\frac{n-1}{2}} \int_{\Sigma} [v^+(y;z) + v^-(y;z)] dA_z \qquad (1$$

• Choose initial data for beam $v^{\pm}(z; z)$ such that

$$Lu_{GB}(y) \sim rac{1}{arepsilon} ilde{g}(y) \delta(
ho(y)) + f_{GE}$$

with $ilde{g} pprox g$.

Theorem (Liu, Ralston, Tanushev, O.R., 2013)

Assume

- Smooth, compactly supported source g(x)
- Index of refraction n(x) smooth and constant for |x| > R
- No trapped rays: $\exists L \ s.t. \ |x(\pm L)| > 2R \ if \ |x(0)| < R, \ |p(0)| = n(x(0))$
- No initial data error $\tilde{g} = g$.

Then with C independent of ε and α ,

$$||\boldsymbol{u}-\boldsymbol{u}_{GB}||_{L^2(|\boldsymbol{x}|<\boldsymbol{R})} \leq C\varepsilon^{K/2},$$

3

*K*th order beams, Helmholtz case

Theorem (Liu, Ralston, Tanushev, O.R., 2013)

Assume

- Smooth, compactly supported source g(x)
- Index of refraction n(x) smooth and constant for |x| > R
- No trapped rays: ∃L s.t. |x(±L)| > 2R if |x(0)| < R, |p(0)| = n(x(0))</p>
- No initial data error $\tilde{g} = g$.

Then with C independent of ε and α ,

$$||\boldsymbol{u} - \boldsymbol{u}_{GB}||_{L^2(|\boldsymbol{x}| < R)} \le C\varepsilon^{K/2},$$

• Superposition in physical space.

3

Kth order beams, Helmholtz case

Theorem (Liu, Ralston, Tanushev, O.R., 2013)

Assume

- Smooth, compactly supported source g(x)
- Index of refraction n(x) smooth and constant for |x| > R
- No trapped rays: ∃L s.t. |x(±L)| > 2R if |x(0)| < R, |p(0)| = n(x(0))</p>
- No initial data error $\tilde{g} = g$.

Then with C independent of ε and α ,

$$||\boldsymbol{u} - \boldsymbol{u}_{GB}||_{L^2(|\boldsymbol{x}| < R)} \le C\varepsilon^{K/2},$$

- Superposition in physical space.
- Convergence of all beams independent of dimension and presence of caustics.

э.

• Use energy estimate

$$\|u_{GB}(t,\cdot) - u(t,\cdot)\|_{E} \leq \|u_{GB}(0,\cdot) - u(0,\cdot)\|_{E} + C\varepsilon \sup_{t \in [0,T]} \|\Box u_{GB}(t,\cdot)\|_{L^{2}},$$

크

Sketch of proof, wave equation

Use energy estimate

$$\|u_{GB}(t,\cdot) - u(t,\cdot)\|_{E} \le \|u_{GB}(0,\cdot) - u(0,\cdot)\|_{E} + C\varepsilon \sup_{t \in [0,T]} \|\Box u_{GB}(t,\cdot)\|_{L^{2}},$$

The residual is of the form

$$\Box u_{GB}(t, \mathbf{y}) = \varepsilon^{K/2-q} \sum_{j=1}^{J} \varepsilon^{r_j} \mathcal{T}_j^{\varepsilon}[f_j](t, \mathbf{y}) + \mathcal{O}(\varepsilon^{\infty}) ,$$

where $r_j \ge 0$, J finite and $f_j \in L^2$ (all independent of ε). $\mathcal{T}_j^{\varepsilon} : L^2 \to L^2$ belongs to a class of oscillatory integral operators.

Use energy estimate

$$\|u_{GB}(t,\cdot)-u(t,\cdot)\|_{E} \leq \|u_{GB}(0,\cdot)-u(0,\cdot)\|_{E} + C\varepsilon \sup_{t\in[0,T]} \|\Box u_{GB}(t,\cdot)\|_{L^{2}},$$

• The residual is of the form

$$\Box u_{GB}(t, y) = \varepsilon^{K/2-q} \sum_{j=1}^{J} \varepsilon^{r_j} \mathcal{T}_j^{\varepsilon}[f_j](t, y) + \mathcal{O}(\varepsilon^{\infty}) ,$$

where $r_j \ge 0$, J finite and $f_j \in L^2$ (all independent of ε). $\mathcal{T}_j^{\varepsilon} : L^2 \to L^2$ belongs to a class of oscillatory integral operators.

• Together we get (if initial data exact)

$$\|u_{GB}(t,\cdot)-u(t,\cdot)\|_{E} \leq C(T)\varepsilon^{K/2}\sum_{j=1}^{J}\varepsilon^{r_{j}}\|\mathcal{T}_{j}^{\varepsilon}\|_{L^{2}}\|f_{j}\|_{L^{2}}+\mathcal{O}(\varepsilon^{\infty})$$

Sketch of proof, cont.

We have

$$\|u_{GB}(t,\cdot)-u(t,\cdot)\|_{E} \leq C(T)\varepsilon^{K/2}\sum_{j=1}^{J}\|\mathcal{T}_{j}^{\varepsilon}\|_{L^{2}}+\mathcal{O}(\varepsilon^{\infty})$$

where, in its simplest form,

$$\mathcal{T}^{\varepsilon}[w](t,y) := \varepsilon^{-\frac{n+|\alpha|}{2}} \int_{\mathcal{K}_0} w(z)(y-x(t;z))^{\alpha} e^{i\phi(t,y-x(t;z);z)/\varepsilon} dz,$$

for some multi-index α , Gaussian beam phase ϕ and geometrical optics rays x(t; z) with x(0; z) = z.

Sketch of proof, cont.

We have

$$\|u_{GB}(t,\cdot)-u(t,\cdot)\|_{E} \leq C(T)\varepsilon^{K/2}\sum_{j=1}^{J}\|\mathcal{T}_{j}^{\varepsilon}\|_{L^{2}}+\mathcal{O}(\varepsilon^{\infty})$$

where, in its simplest form,

$$\mathcal{T}^{\varepsilon}[w](t,y) := \varepsilon^{-\frac{n+|\alpha|}{2}} \int_{\mathcal{K}_0} w(z)(y-x(t;z))^{\alpha} e^{i\phi(t,y-x(t;z);z)/\varepsilon} dz,$$

for some multi-index α , Gaussian beam phase ϕ and geometrical optics rays x(t; z) with x(0; z) = z.

Result follows if we prove that T^{ε} is bounded in L^2 independent of ε ,

$$||\mathcal{T}^{\varepsilon}||_{L^2} \leq C.$$

This is the key estimate of our proof.

Olof Runborg (KTH)

A B A B A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A
 A

Sketch of proof, cont.

Estimate of $||\mathcal{T}^{\varepsilon}||_{L^2}$, where

$$\mathcal{T}^{\varepsilon}[w](t,y) := \varepsilon^{-\frac{n+|\alpha|}{2}} \int_{\mathcal{K}_0} w(z)(y-x(t;z))^{\alpha} e^{i\phi(t,y-x(t;z);z)/\varepsilon} dz.$$

Main difficulty: no globally invertible map x(0; z) = z → x(t; z) because of caustics.

э
Sketch of proof, cont.

Estimate of $||\mathcal{T}^{\varepsilon}||_{L^2}$, where

$$\mathcal{T}^{\varepsilon}[w](t,y) := \varepsilon^{-\frac{n+|\alpha|}{2}} \int_{\mathcal{K}_0} w(z)(y-x(t;z))^{\alpha} e^{i\phi(t,y-x(t;z);z)/\varepsilon} dz.$$

- Main difficulty: no globally invertible map x(0; z) = z → x(t; z) because of caustics.
- Mapping (x(0; z), p(0; z)) → (x(t; z), p(t; z) is however globally invertible and smooth. Gives the "non-squeezing" property,

$$|z-z'| \le |p(t;z) - p(t;z')| + |x(t;z) - x(t;z')| \le c_2|z-z'|$$

Sketch of proof, cont.

Estimate of $||\mathcal{T}^{\varepsilon}||_{L^2}$, where

$$\mathcal{T}^{\varepsilon}[w](t,y) := \varepsilon^{-\frac{n+|\alpha|}{2}} \int_{\mathcal{K}_0} w(z)(y-x(t;z))^{\alpha} e^{i\phi(t,y-x(t;z);z)/\varepsilon} dz.$$

- Main difficulty: no globally invertible map x(0; z) = z → x(t; z) because of caustics.
- Mapping (x(0; z), p(0; z)) → (x(t; z), p(t; z) is however globally invertible and smooth. Gives the "non-squeezing" property,

$$|z-z'| \le |p(t;z)-p(t;z')| + |x(t;z)-x(t;z')| \le c_2|z-z'|$$

 Allows us to use stationary phase arguments close to caustics, and carefully control cancellations of oscillations there (similar to [Swart,Rousse], [Bougacha, Akian, Alexandre]).

1

3

The estimate

$$||u(t,\cdot) - u_{GB}(t,\cdot)||_E \leq O(\varepsilon^{K/2})$$

is sharp for individual beams (relative error). But for superpositions?

The estimate

$$||u(t,\cdot) - u_{GB}(t,\cdot)||_E \le O(\varepsilon^{K/2})$$

is sharp for individual beams (relative error). But for superpositions?

 Predicts convergence rate of first order beam to be only O(√ε). These beams are based on same high frequency approximation as geometrical optics which has O(ε) accuracy.

The estimate

$$||u(t,\cdot) - u_{GB}(t,\cdot)||_E \le O(\varepsilon^{K/2})$$

is sharp for individual beams (relative error). But for superpositions?

- Predicts convergence rate of first order beam to be only O(√ε). These beams are based on same high frequency approximation as geometrical optics which has O(ε) accuracy.
- Numerical experiments suggests a better rate for odd order beams

The estimate

$$||u(t,\cdot) - u_{GB}(t,\cdot)||_E \leq O(\varepsilon^{K/2})$$

is sharp for individual beams (relative error). But for superpositions?

- Predicts convergence rate of first order beam to be only O(√ε). These beams are based on same high frequency approximation as geometrical optics which has O(ε) accuracy.
- Numerical experiments suggests a better rate for odd order beams
- For the Helmholtz case we have proved [Motamed, OR] that

$$|u(x) - u_{GB(x)}| \le O(\varepsilon^{\lceil K/2 \rceil})$$

for the Taylor expansion part of the error away from caustics. This gives $O(\varepsilon)$ for first order beams.

The estimate

$$||u(t,\cdot) - u_{GB}(t,\cdot)||_E \leq O(\varepsilon^{K/2})$$

is sharp for individual beams (relative error). But for superpositions?

- Predicts convergence rate of first order beam to be only O(√ε). These beams are based on same high frequency approximation as geometrical optics which has O(ε) accuracy.
- Numerical experiments suggests a better rate for odd order beams
- For the Helmholtz case we have proved [Motamed, OR] that

$$|u(x) - u_{GB(x)}| \le O(\varepsilon^{\lceil K/2 \rceil})$$

for the Taylor expansion part of the error away from caustics. This gives $O(\varepsilon)$ for first order beams.

More error cancellations coming in for odd order beams?
(⇒ no gain in using even order beams)

Olof Runborg (KTH)

Cusp caustic

Consider the test case where

$$\Phi(0, y) = -y_1 + y_2^2,$$

 $A(0, y) = e^{-10|y|^2}.$

- Cusp caustic at t = 0.5
- Two fold caustics at t > 0.5



< A

Cusp caustic

Consider the test case where

$$\Phi(0, y) = -y_1 + y_2^2,$$

 $A(0, y) = e^{-10|y|^2}.$

- Cusp caustic at t = 0.5
- Two fold caustics at t > 0.5



< A

Cusp caustic

Consider the test case where

$$\Phi(0, y) = -y_1 + y_2^2,$$

 $A(0, y) = e^{-10|y|^2}.$

- Cusp caustic at t = 0.5
- Two fold caustics at t > 0.5



< A

Cusp caustic

Consider the test case where

$$\Phi(0, y) = -y_1 + y_2^2,$$

 $A(0, y) = e^{-10|y|^2}.$

- Cusp caustic at *t* = 0.5
- Two fold caustics at *t* > 0.5



∃ >

Cusp caustic

$$\Phi(0, y) = -y_1 + y_2^2,$$

 $A(0, y) = e^{-10|y|^2}.$

- Cusp caustic at t = 0.5
- Two fold caustics at t > 0.5



Cusp caustic

$$\Phi(0, y) = -y_1 + y_2^2,$$

 $A(0, y) = e^{-10|y|^2}.$

- Cusp caustic at t = 0.5
- Two fold caustics at t > 0.5



Cusp caustic

Consider the test case where

$$\Phi(0, y) = -y_1 + y_2^2,$$

 $A(0, y) = e^{-10|y|^2}.$

- Cusp caustic at t = 0.5
- Two fold caustics at t > 0.5



э.

Cusp caustic

$$\Phi(0, y) = -y_1 + y_2^2,$$

 $A(0, y) = e^{-10|y|^2}.$

- Cusp caustic at t = 0.5
- Two fold caustics at t > 0.5



Cusp caustic

$$\Phi(0, y) = -y_1 + y_2^2,$$

 $A(0, y) = e^{-10|y|^2}.$

- Cusp caustic at t = 0.5
- Two fold caustics at t > 0.5



Cusp caustic

Consider the test case where

$$\Phi(0, y) = -y_1 + y_2^2,$$

 $A(0, y) = e^{-10|y|^2}.$

- Cusp caustic at *t* = 0.5
- Two fold caustics at *t* > 0.5



< A

э.

Cusp caustic

Consider the test case where

$$\Phi(0, y) = -y_1 + y_2^2,$$

 $A(0, y) = e^{-10|y|^2}.$

- Cusp caustic at *t* = 0.5
- Two fold caustics at *t* > 0.5



< A

∃ >

Cusp caustic, convergence



Olof Runborg (KTH)

Gaussian Beam Approximation

IPP Garching, 2013 36 / 36

Cusp caustic, convergence

