Development of a stable coupling of the Yee scheme with linear current

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Outline

1 Motivation

2 Stable schemes

3 Explicit schemes

4 Numerical results and perspectives
The equations

No magnetisation ($\mu = \mu_0$).
Maxwell equations with a linear current derive from the linearization $\mu_0 |H| \ll |B_0|$ of the Vlasov-Maxwell system (for electrons) around a strong magnetic field $B_0$:

$$
\begin{align*}
-\varepsilon_0 \partial_t E + \nabla \wedge H &= -q_e N_e(\mathbf{x}) u_e, \\
\mu_0 \partial_t H + \nabla \wedge E &= 0, \\
m_e \partial_t u_e &= -q_e (E + B_0(\mathbf{x}) \wedge u_e) - \nu m_e u_e.
\end{align*}
$$

Or, writing $J = -q_e N_e(\mathbf{x}) u_e$,

$$
\begin{align*}
\varepsilon_0 \partial_t E &= \nabla \wedge H - J, \\
\mu_0 \partial_t H &= -\nabla \wedge E, \\
\partial_t J &= \varepsilon_0 \omega_p^2 E + \omega_c b \wedge J
\end{align*}
$$

with $\omega_p(\mathbf{x}) = \sqrt{\frac{q_e^2 N_e(\mathbf{x})}{m \varepsilon_0}}$, $\omega_c(\mathbf{x}) = \frac{q_e |B_0(\mathbf{x})|}{m_e}$ and $b(\mathbf{x}) = -\frac{B_0(\mathbf{x})}{|B_0(\mathbf{x})|}$. 

Munich: Pereverzev legacy 13/10/2013
Direct simulation of reflectometry configuration

The domain is a parallelepiped (≈ 1500 cells in x direction) with an antenna on the side: pulsation $\omega$

- Cut-off: in O mode (TM), waves propagate if $\omega \geq \omega_p(x)$.
- Cyclotron resonance: $\omega = \omega_c$
- Hybrid resonance: $\omega^2 = \omega_p(x)^2 + \omega_c^2$

Based on the Yee scheme for the \((E, H)\) field: general form is

\[
\begin{align*}
\frac{\varepsilon_0}{\Delta t}(E_{n+1}^n - E^n) &= RH_{n+\frac{1}{2}}^n - J_{n+\frac{1}{2}}^n \\
\frac{\mu_0}{\Delta t}(H_{n+\frac{3}{2}}^n - H_{n+\frac{1}{2}}^n) &= -R^t E_{n+1}^n \\
\frac{1}{\Delta t}(J_{n+\frac{3}{2}}^n - J_{n+\frac{1}{2}}^n) &= \varepsilon_0 \omega_p^2 E_{n+1}^n + \omega_c b \wedge \frac{1}{2}(J_{n+\frac{3}{2}}^n + J_{n+\frac{1}{2}}^n).
\end{align*}
\]

Need to specify the operator \(\wedge_h\) on the Yee grid
X-mode equations

X-mode = Transverse electric (O-mode not discussed in this talk).

\[
\begin{align*}
-\varepsilon_0 \partial_t E_x + \partial_y H_z &= J_x, \quad J_x = eN_e u_x, \\
\varepsilon_0 \partial_t E_y - \partial_x H_z &= J_y, \quad J_y = eN_e u_y, \\
\mu_0 \partial_t H_z + \partial_x E_y - \partial_y E_x &= 0, \\
m_e \partial_t u_x &= eE_x + eu_y B_z^0, \\
m_e \partial_t u_y &= eE_y - eu_x B_z^0.
\end{align*}
\]

Call VLC external
Main difficulty

In fusion plasmas, $N_e(x)$ has huge fluctuations along the main axis.

Negative density (num. or measurement artifact) induces automatically an instability, as well as strong spatial gradient at the plasma edge (phys.) or inside the plasma (phys.).
Example of unstability (for large times)

Magnetic field $20 \log_{10} |H_z|$ (where $|H_z| = \|H_z\|_{L^\infty}$) vs. time step and level of fluctuations

Expertise from F. Da Silva and S. Heuraux.
Some references

- Xu-Yuan, FDTD Formulations for Scattering From 3-D Anisotropic Magnetized Plasma Objects, IEEE-2006
- An unconditionally stable (?) time-domain discretization on cartesian meshes for the simulation of nonuniform magnetized cold plasma, JCP-2012, Tierens-Zutter
- Smithe, Finite-difference time-domain simulation of fusion plasmas at radiofrequency time scales, Physics of Plasmas, 2007

Energy conservation at the continuous level

- For simplicity: constant density profile $N_e(x, t) = N_e(x)$.
- Inside the computational domain (no boundaries assumed), the total energy is conserved in time,

$$
\frac{d}{dt} \int_{\Omega} \left( \frac{\epsilon_0 |E|^2}{2} + \frac{|H|^2}{2\mu_0} + \frac{m_e N_e(x)|ue|^2}{2} \right) dv = 0.
$$

- Using "normalized" variables $\hat{E} := \frac{1}{c} E$, $\hat{H} := \mu_0 H$ and $\hat{J} := \frac{1}{\omega_p c \epsilon_0} J$, we have

$$
\frac{d}{dt} \int \left( \frac{\hat{E}^2}{2} + \frac{\hat{H}^2}{2} + \frac{\hat{J}^2}{2} \right) dv = 0.
$$
Classical stability analysis for the Yee scheme

With normalized variables, the Yee scheme \((J = 0)\) reads

\[
\begin{cases}
\frac{1}{\Delta t} (\hat{E}^{n+1} - \hat{E}^n) = cR \hat{H}^{n+\frac{1}{2}} \\
\frac{1}{\Delta t} (\hat{H}^{n+\frac{1}{2}} - \hat{H}^{n-\frac{1}{2}}) = -cR^t \hat{E}^n
\end{cases}
\]

where \(\hat{E} := \frac{1}{c} E\)

\(\hat{H} := \mu_0 H\).

In particular, the energy \(\hat{E}^n := \|\hat{E}^n\|^2_h + \|\hat{H}^{n-\frac{1}{2}}\|^2_h\) satisfies

\[
\hat{E}^{n+1} - \hat{E}^n = c\Delta t(\langle R \hat{H}^{n+\frac{1}{2}}, \hat{E}^{n+1} + \hat{E}^n \rangle - \langle R^t \hat{E}^n, \hat{H}^{n+\frac{1}{2}} + \hat{H}^{n-\frac{1}{2}} \rangle)
\]

hence \(E^n := \hat{E}^n - c\Delta t \langle \hat{E}^n, R \hat{H}^{n-\frac{1}{2}} \rangle\) is constant. Moreover,

\[
|\langle \hat{E}^n, R \hat{H}^{n-\frac{1}{2}} \rangle| \leq \frac{1}{2} \|R\| \hat{E}^n \implies \hat{E}^n (1 - \frac{c\Delta t}{2} \|R\|) \leq E^n
\]

\[
\implies \text{Stability in the energy norm : for } c\Delta t < 2/\|R\| = h/\sqrt{3}
\]
Stability analysis for an abstract Yee+J scheme

With \( \hat{E}, \hat{H} \) and \( \hat{J} := \frac{1}{\omega_p c \varepsilon_0} J \), the “abstract” Yee+J scheme is

\[
\begin{align*}
\frac{1}{\Delta t} (\hat{E}^{n+1} - \hat{E}^n) &= cR \hat{H}^{n+\frac{1}{2}} - \omega_p \hat{J}^{n+\frac{1}{2}} \\
\frac{1}{\Delta t} (\hat{H}^{n+\frac{1}{2}} - \hat{H}^{n-\frac{1}{2}}) &= -cR^t \hat{E}^n \\
\frac{1}{\Delta t} (\hat{J}^{n+\frac{1}{2}} - \hat{J}^{n-\frac{1}{2}}) &= \omega_p \hat{E}^n + \omega_c b \wedge_h \frac{\hat{J}^{n+\frac{1}{2}} + \hat{J}^{n-\frac{1}{2}}}{2}
\end{align*}
\]

Here the energy \( \hat{E}^n := \| \hat{E}^n \|^2 + \| \hat{H}^{n-\frac{1}{2}} \|^2 + \| \hat{J}^{n-\frac{1}{2}} \|^2 \) satisfies

\[-\Delta t (\langle \omega_p \hat{J}^{n+\frac{1}{2}}, \hat{E}^{n+1} + \hat{E}^n \rangle - \langle \omega_p \hat{E}^n, \hat{J}^{n+\frac{1}{2}} + \hat{J}^{n-\frac{1}{2}} \rangle)
\]

provided \( \langle V, b \wedge h V \rangle = 0 \) for all \( V \).

Stability in the energy norm: for \( \frac{\Delta t}{2} \left( \frac{12c^2}{h^2} + \| \omega_p \|_{L^\infty} \right)^{\frac{1}{2}} < 1 \).
Remark on average cross products

- One can use local averages to define a 2nd order cross product,

\[
(b \wedge_h V)_x := b_y \{ V_z \} - b_z \{ V_y \}
\]
\[
(b \wedge_h V)_y := \cdots
\]
\[
(b \wedge_h V)_z := \cdots
\]

Then if \( b(x) = -\frac{B_0(x)}{|B_0|} \) is uniform,

\[
\langle V, b \wedge_h V \rangle = 0 \quad \text{holds for all } V
\]

→ previous analysis applies.

- If \( b(x) \) is not uniform this is not so clear...
Improved stability for a new Yee+J scheme

Discretizing the current on \( t_n, t_{n+1}, \ldots \) yields a new scheme

\[
\begin{align*}
\frac{1}{\Delta t}(\hat{E}^{n+1} - \hat{E}^n) &= cR \hat{H}^{n+\frac{1}{2}} - \omega_p \frac{\hat{j}^{n+1} + \hat{j}^n}{2} \\
\frac{1}{\Delta t}(\hat{H}^{n+\frac{1}{2}} - \hat{H}^{n-\frac{1}{2}}) &= -cR^t \hat{E}^n \\
\frac{1}{\Delta t}(\hat{j}^{n+1} - \hat{j}^{n-1}) &= \omega_p \{\hat{E}\}^{n+\frac{1}{2}} + \omega_c b \wedge h \frac{\hat{j}^{n+1} + \hat{j}^n}{2}.
\end{align*}
\]

The energy \( \hat{E}^n \) satisfies

\[
\hat{E}^{n+1} - \hat{E}^n = c\Delta t \left( \langle R \hat{H}^{n+\frac{1}{2}}, \hat{E}^{n+1} + \hat{E}^n \rangle - \langle R^t \hat{E}^n, \hat{H}^{n+\frac{1}{2}} + \hat{H}^{n-\frac{1}{2}} \rangle \right)
\]

\[
- \Delta t \left( \langle \omega_p \{\hat{j}\}^{n+\frac{1}{2}}, 2\{\hat{E}\}^{n+\frac{1}{2}} \rangle - \langle \omega_p \{\hat{E}\}^{n+\frac{1}{2}}, 2\{\hat{j}\}^{n+\frac{1}{2}} \rangle \right)
\]

once again provided \( \langle V, b \wedge h V \rangle = 0 \) for all \( V \).

Stability in the energy norm: for \( c\Delta t < 2/\|R\| = h/\sqrt{3} \).
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Example of the Xu-Yuan scheme

Based on the Yee scheme for the \((E, H)\) field: general form is

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\frac{1}{\Delta t} (J^{n+\frac{3}{2}} - J^{n+\frac{1}{2}}) &= \varepsilon_0 \omega_p^2 E^{n+1} + \omega_c b \wedge \frac{1}{2} (J^{n+\frac{3}{2}} + J^{n+\frac{1}{2}}).
\end{align*}
\]
Example of the Xu-Yuan scheme

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\frac{1}{\Delta t} (J^{n+\frac{3}{2}} - J^{n+\frac{1}{2}}) &= \varepsilon_0 \omega_p^2 E^{n+1} + \omega_c b \wedge h \frac{1}{2} (J^{n+\frac{3}{2}} + J^{n+\frac{1}{2}}).
\end{align*}
\]

Need an explicit solver with the \(b \wedge h\) operator.
Problem with the X-Y approach

Consider once again the cross product by local averages

\[
(b \wedge_h V)_x := b_y \{ V_z \} - b_z \{ V_y \}
\]

\[
(b \wedge_h V)_y := \cdots
\]

\[
(b \wedge_h V)_z := \cdots
\]

The result

\[
\begin{aligned}
\frac{\varepsilon_0}{\Delta t} (E^{n+1} - E^n) &= RH^{n+\frac{1}{2}} - J^{n+\frac{1}{2}} \\
\frac{\mu_0}{\Delta t} (H^{n+\frac{3}{2}} - H^{n+\frac{1}{2}}) &= -R^t E^{n+1} \\
\frac{1}{\Delta t} (J^{n+\frac{3}{2}} - J^{n+\frac{1}{2}}) &= \varepsilon_0 \omega_p^2 E^{n+1} + \omega_c b \wedge_h \frac{1}{2} (J^{n+\frac{3}{2}} + J^{n+\frac{1}{2}}).
\end{aligned}
\]

is a global scheme which needs a linear solver to invert the matrice.
Solution: use clustered cross-products

Instead, choose a pattern \((\alpha, \beta, \gamma) \in \{-1, +1\}^3\) and define the first order cross product with local clusters:

\[
\begin{align*}
(b \wedge_h V)_x |_{i, j, k} &= b_y V_z |_{i, j, k + \frac{\gamma}{2}} - b_z \{ V_y \} |_{i, j + \frac{\beta}{2}, k} \\
(b \wedge_h V)_y |_{i, j + \frac{\beta}{2}, k} &= b_z V_x |_{i, j, k} - b_x \{ V_z \} |_{i, j, k + \frac{\gamma}{2}} \\
(b \wedge_h V)_z |_{i, j, k + \frac{\gamma}{2}} &= b_x V_y |_{i, j, k} - b_y \{ V_x \} |_{i + \frac{\alpha}{2}, j, k}
\end{align*}
\]

The resulting scheme

\[
\begin{align*}
\frac{\varepsilon_0}{\Delta t} (E^{n+1} - E^n) &= RH^{n+\frac{1}{2}} - J^{n+\frac{1}{2}} \\
\frac{\mu_0}{\Delta t} (H^{n+\frac{3}{2}} - H^{n+\frac{1}{2}}) &= -R^t E^{n+1} \\
\frac{1}{\Delta t} (J^{n+\frac{3}{2}} - J^{n+\frac{1}{2}}) &= \varepsilon_0 \omega_p^2 E^{n+1} + \omega_c b \wedge_h \frac{1}{2} (J^{n+\frac{3}{2}} + J^{n+\frac{1}{2}}).
\end{align*}
\]

Can be solved with a local procedure (i.e. solution is explicit and local).
Abstract criterion

The criterion for explicit scheme writes: \((b \wedge h)^4 = -(b \wedge h)^2\).
Indeed one has the implications

\[ J - \alpha b \wedge h J = Z, \]

\[ J - \alpha^2 (b \wedge h)^2 J = (I + \alpha b \wedge h)Z, \]

\[ (1 + \alpha^2)(b \wedge h)^2 J = (b \wedge h)^2(I + \alpha b \wedge h)Z, \]

\[ J = (I + \alpha b \wedge h)Z + \frac{\alpha^2}{1 + \alpha^2}(b \wedge h)^2(I + \alpha b \wedge h)Z. \]

This algebra is enough to compute the solution by means of explicit and local formulas (for MXYK and new Kernel).
The coupling of the Yee scheme and a linear current is

- Stable for: \( (V, b \wedge_h V) = 0 \)
- Explicit for: \( (b \wedge_h)^4 = -(b \wedge_h)^2 \)

Solution (so far) is clustered first order product

- Additional and natural condition is that \( \omega_p \) and \( \omega_c \) are the same within a cluster.
Cut of the electronic density in the horizontal direction. An additional kink (in red) is sometimes added at $x = 500$ to evaluate the effect of an extremely strong gradient.
With the kick and 30% noise

An instability shows up near $x = 500$ cells on the left, near $x = 1000$

Without the kick but 40% noise
Motivation

Stable schemes

Explicit schemes

Numerical results and perspectives

- With respect to the time and to the level of noise.
- With the kick on the left, without the kick on the right.
With the first order vectorial product

\[ 20 \log_{10} \| H_z \|_{L^\infty}, \] with respect to the time and to the level of noise. The computation is done

We observe unconditional stability, with however more amplitude for a higher level of noise. The number of time steps is much greater than in previous figure to illustrate the long time stability of the method.
Energy dissipation

Initial data is a Dirac mass, at the exact foot of the electronic density ramp. The external magnetic field used in this set of runs was $B_0 = 0.95 \, T$. The plasma density $N_e(x)$ is linear, with its edge at $x = 500$ grid point. The number of iterations considered is $N = 700$ (far from PML layer).

(slide courtesy of F. Da Silva)
Need to use clustered multiplications by scalar fields, consistent with clustered cross products.

Counter-intuitive: the stable and explicit scheme is globally first order (and not second order like the standard Yee scheme).

Possibility to average in time by alternating the cluster patterns \((\alpha, \beta, \gamma)\) in \((-1, +1)^3\).

Work in progress for direct simulation of time-dependent densities \(N_e = N_e(x, t)\) (Doppler reflectometry).

A paper is being written.
Motivation

Stable schemes

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Numerical results and perspectives

An open question (is it really ?)

Look at

\[
\begin{cases}
-\varepsilon_0 \partial_t E + \nabla \land H = -q_e N_e(x) u_e, \\
\mu_0 \partial_t H + \nabla \land E = 0, \\
m_e \partial_t u_e = -q_e (E + B_0(x) \land u_e) - \nu m_e u_e
\end{cases}
\]

plus harmonic forcing on the boundary, plus initial condition, plus friction $\nu > 0$.

Assume resonance configuration (cyclotron, hybrid, . . . ) : do we have

\[
\lim_{\nu \to 0^+} \lim_{T \to \infty} = \lim_{T \to \infty} \lim_{\nu \to 0^+} ?
\]

In other words, do we have Limit absorption$=$Limit amplitude ?

If not, which one is the correct physical solution ?
Motivation

Stable schemes

Explicit schemes

Numerical results and perspectives

\[
\lim_{\nu \to 0^+} \lim_{T \to \infty} (L.M. \text{ Imbert-Grard})
\]