Development of a stable coupling of the Yee scheme with linear current

Martin Campos Pinto (LJLL), Bruno Després (LJLL)
Stéphane Heuraux (IJL), Filipe Da Silva (IPFN+IST)

15 octobre 2013

Outline

The equations

No magnetisation ($\mu = \mu_0$).
Maxwell equations with a linear current derive from the linearization $\mu_0|H| \ll |B_0|$ of the Vlasov-Maxwell system (for electrons) around a strong magnetic field B_0 :

$$\left\{ \begin{array}{l} -\varepsilon_0 \partial_t E + \nabla \wedge H = -q_e N_e(\mathbf{x}) u_e, \\ \mu_0 \partial_t H + \nabla \wedge E = 0, \\ m_e \partial_t u_e = -q_e(E + B_0(\mathbf{x}) \wedge u_e) - \cancel{\nu m_e u_e}. \end{array} \right.$$

Or, writing $J = -q_e N_e(\mathbf{x}) u_e$,

$$\left\{ \begin{array}{l} \varepsilon_0 \partial_t E = \nabla \wedge H - J, \\ \mu_0 \partial_t H = -\nabla \wedge E, \\ \partial_t J = \varepsilon_0 \omega_p^2 E + \omega_c b \wedge J \end{array} \right.$$

with $\omega_p(\mathbf{x}) = \sqrt{\frac{q_e^2 N_e(\mathbf{x})}{m \varepsilon_0}}$, $\omega_c(\mathbf{x}) = \frac{q_e |B_0(\mathbf{x})|}{m_e}$ and $b(\mathbf{x}) = -\frac{B_0(\mathbf{x})}{|B_0(\mathbf{x})|}$.

Direct simulation of reflectometry configuration

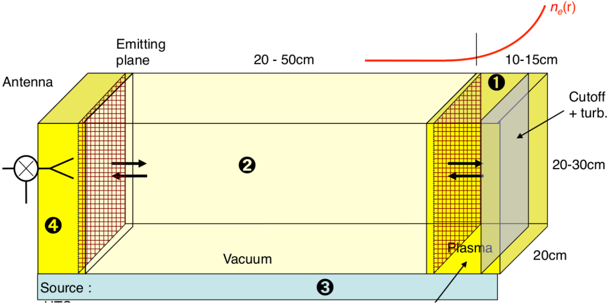
Motivation

Stable schemes

Explicit schemes

Numerical results and perspectives

The domain is a parallepiped ($\approx$ 1500 cells in x direction) with an antenna on the side : pulsation ω



- Cut-off : in O mode (TM), waves propagate if $\omega \geq \omega_p(\mathbf{x})$.
- Cyclotron resonance : $\omega = \omega_c$
- Hybrid resonance : $\omega^2 = \omega_p(\mathbf{x})^2 + \omega_c^2$

"Standard" scheme of Xu-Yuan (2006)

Motivation

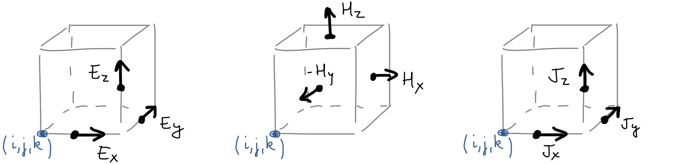
Stable schemes

Explicit schemes

Numerical results and perspectives

Based on the Yee scheme for the (E, H) field : general form is

$$\begin{cases} \frac{\varepsilon_0}{\Delta t}(E^{n+1} - E^n) = RH^{n+\frac{1}{2}} - J^{n+\frac{1}{2}} \\ \frac{\mu_0}{\Delta t}(H^{n+\frac{3}{2}} - H^{n+\frac{1}{2}}) = -R^t E^{n+1} \\ \frac{1}{\Delta t}(J^{n+\frac{3}{2}} - J^{n+\frac{1}{2}}) = \varepsilon_0 \omega_p^2 E^{n+1} + \omega_c b \wedge \frac{1}{2}(J^{n+\frac{3}{2}} + J^{n+\frac{1}{2}}). \end{cases}$$



- $\rightarrow$ Need to specify the operator "$\wedge_h$" on the Yee grid

X-mode equations

Motivation

Stable schemes

Explicit schemes

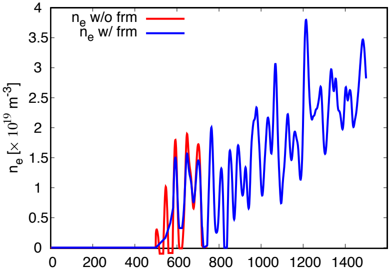
Numerical results and perspectives

X-mode=Transverse electric (O-mode not discussed in this talk).

$$
\begin{cases}
-\varepsilon_0 \partial_t E_x + \partial_y H_z & = J_x, \qquad J_x = eN_e u_x, \\
\varepsilon_0 \partial_t E_y - \partial_x H_z & = J_y, \qquad J_y = eN_e u_y, \\
\\
\mu_0 \partial_t H_z + \partial_x E_y - \partial_y E_x & = 0, \\
\\
m_e \partial_t u_x & = eE_x + eu_y B_z^0, \\
m_e \partial_t u_y & = eE_y - eu_x B_z^0.
\end{cases}
$$

Call VLC external

In fusion plasmas, $N_e(\mathbf{x})$ has huge fluctuations along the main axis



Negative density (**num. or measurement artifact**) induces automatically an instability, as well as strong spatial gradient at the plasma edge (**phys.**) or inside the plasma (**phys.**).

Example of unstability (for large times)

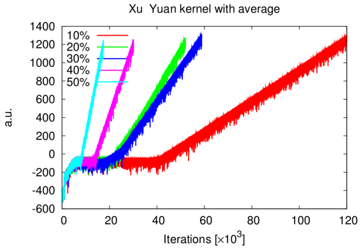
Motivation

Stable schemes

Explicit schemes

Numerical results and perspectives

Magnetic field $20\log_{10}|H_z|$ (where $|H_z| = \|H_z\|_{L^\infty}$) vs. time step and level of fluctuations



Expertise from F. Da Silva and S. Heuraux.

Some references

Motivation

Stable schemes

Explicit schemes

Numerical results and perspectives

• Xu-Yuan, FDTD Formulations for Scattering From 3-D Anisotropic Magnetized Plasma Objects, IEEE-2006

• Bohner, Simulation of Microwave Propagation in a Fusion Plasma, Diploma Thesis, MaxPlanck, 2011

• An unconditionally stable (?) time-domain discretization on cartesian meshes for the simulation of nonuniform magnetized cold plasma, JCP-2012, Tierens-Zutter

• Yu-Simpson, An E-J Collocated 3-D FDTD Model of Electromagnetic Wave Propagation in Magnetized Cold Plasma, Ieee Transations on antennas and propagation, 58-2, 2010

• Smithe, Finite-difference time-domain simulation of fusion plasmas at radiofrequency time scales, Physics of plasmas, 2007

• da Silva-Heuraux-Ribeiro-Scott, (2013). Development of a 2D full-wave JE-FDTD Maxwell X-mode code for reflectometry simulation (pp. 16). Presented at the 9th International Reflectometry Workshop.

Motivation

Stable schemes

Explicit schemes

Numerical results and perspectives

Outline

Energy conservation at the continuous level

- For simplicity : constant density profile $N_e(\mathbf{x}, t) = N_e(\mathbf{x})$.
- Inside the computational domain (no boundaries assumed), the total energy is conserved in time,

$$\frac{d}{dt}\int_{\Omega}\left(\frac{\varepsilon_0|E|^2}{2}+\frac{|H|^2}{2\mu_0}+\frac{m_e N_e(\mathbf{x})|u_e|^2}{2}\right)dv = 0.$$

- Using "normalized" variables $\hat{E} := \frac{1}{c}E$, $\hat{H} := \mu_0 H$ and $\hat{J} := \frac{1}{\omega_p c \varepsilon_0}J$, we have

$$\frac{d}{dt}\int\left(\frac{|\widehat{E}|^2}{2}+\frac{|\widehat{H}|^2}{2}+\frac{|\widehat{J}|^2}{2}\right)dv = 0.$$

Classical stability analysis for the Yee scheme

Motivation

Stable schemes

Explicit schemes

Numerical results and perspectives

With normalized variables, the Yee scheme ($J = 0$) reads

$$\begin{cases} \frac{1}{\Delta t}(\hat{E}^{n+1} - \hat{E}^n) = cR\hat{H}^{n+\frac{1}{2}} \\ \frac{1}{\Delta t}(\hat{H}^{n+\frac{1}{2}} - \hat{H}^{n-\frac{1}{2}}) = -cR^t\hat{E}^n \end{cases} \quad \text{where} \quad \begin{cases} \hat{E} := \frac{1}{c}E \\ \hat{H} := \mu_0 H. \end{cases}$$

In particular, the energy $\hat{\mathcal{E}}^n := \|\hat{E}^n\|_h^2 + \|\hat{H}^{n-\frac{1}{2}}\|_h^2$ satisfies

$$\hat{\mathcal{E}}^{n+1} - \hat{\mathcal{E}}^n = c\Delta t\Big(\langle R\hat{H}^{n+\frac{1}{2}}, \hat{E}^{n+1} + \cancel{\hat{E}^n}\rangle - \langle R^t\hat{E}^n, \cancel{\hat{H}^{n+\frac{1}{2}}} + \hat{H}^{n-\frac{1}{2}}\rangle\Big)$$

hence $\mathcal{E}^n := \hat{\mathcal{E}}^n - c\Delta t\langle\hat{E}^n, R\hat{H}^{n-\frac{1}{2}}\rangle$ is constant. Moreover,

$$|\langle\hat{E}^n, R\hat{H}^{n-\frac{1}{2}}\rangle| \le \frac{1}{2}\|R\|\hat{\mathcal{E}}^n \implies \hat{\mathcal{E}}^n(1 - \tfrac{c\Delta t}{2}\|R\|) \le \mathcal{E}^n$$

$\implies$ Stability in the energy norm : for $c\Delta t < 2/\|R\| = h/\sqrt{3}$

Stability analysis for an abstract Yee+J scheme

With $\hat{E}$, $\hat{H}$ and $\hat{J} := \frac{1}{\omega_p c \varepsilon_0} J$, the "abstract" Yee+J scheme is

$$\begin{cases} \frac{1}{\Delta t}(\hat{E}^{n+1} - \hat{E}^n) = cR\hat{H}^{n+\frac{1}{2}} - \omega_p \hat{J}^{n+\frac{1}{2}} \\ \frac{1}{\Delta t}(\hat{H}^{n+\frac{1}{2}} - \hat{H}^{n-\frac{1}{2}}) = -cR^t \hat{E}^n \\ \frac{1}{\Delta t}(\hat{J}^{n+\frac{1}{2}} - \hat{J}^{n-\frac{1}{2}}) = \omega_p \hat{E}^n + \omega_c b \wedge_h \frac{\hat{J}^{n+\frac{1}{2}} + \hat{J}^{n-\frac{1}{2}}}{2} \end{cases}$$

Here the energy $\hat{\mathcal{E}}^n := \|\hat{E}^n\|^2 + \|\hat{H}^{n-\frac{1}{2}}\|^2 + \|\hat{J}^{n-\frac{1}{2}}\|^2$ satisfies

$$-\Delta t\Big(\langle \omega_p \hat{J}^{n+\frac{1}{2}}, \hat{E}^{n+1} + \cancel{\hat{E}^n}\rangle - \langle \omega_p \hat{E}^n, \cancel{\hat{J}^{n+\frac{1}{2}}} + \hat{J}^{n-\frac{1}{2}}\rangle\Big)$$

provided $\langle V, b \wedge_h V\rangle = 0$ for all V.

Stability in the energy norm : for $\frac{\Delta t}{2}\left(\frac{12c^2}{h^2} + \|\omega_p\|^2_{L^\infty}\right)^{\frac{1}{2}} < 1.$

Motivation

Stable schemes

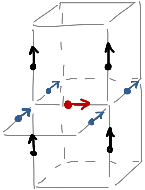
Explicit schemes

Numerical results and perspectives

Remark on average cross products

- One can use local averages to define a 2nd order cross product,

$$(b\wedge_h V)_x := b_y\{V_z\} - b_z\{V_y\}$$
$$(b\wedge_h V)_y := \cdots$$
$$(b\wedge_h V)_z := \cdots$$



- Then if $b(\mathbf{x}) = -\frac{B_0(\mathbf{x})}{|B_0|}$ is uniform,

$$\langle V, b\wedge_h V\rangle = 0 \qquad \text{holds for all } V$$

$\rightarrow$ previous analysis applies.

- If $b(\mathbf{x})$ is not uniform this is not so clear...

Improved stability for a new Yee+J scheme

Discretizing the current on t_n, t_{n+1}, ... yields a new scheme

$$\begin{cases} \frac{1}{\Delta t}(\hat{E}^{n+1} - \hat{E}^n) = cR\hat{H}^{n+\frac{1}{2}} - \omega_p \frac{\widehat{J}^{n+1} + \widehat{J}^n}{2} \\ \frac{1}{\Delta t}(\hat{H}^{n+\frac{1}{2}} - \hat{H}^{n-\frac{1}{2}}) = -cR^t\hat{E}^n \\ \frac{1}{\Delta t}(\hat{J}^{n+1} - \hat{J}^{n-1}) = \omega_p\{\hat{E}\}^{n+\frac{1}{2}} + \omega_c b \wedge_h \frac{\widehat{J}^{n+1} + \widehat{J}^n}{2}. \end{cases}$$

The energy $\hat{\mathcal{E}}^n$ satisfies

$$\hat{\mathcal{E}}^{n+1} - \hat{\mathcal{E}}^n = c\Delta t\Big(\langle R\hat{H}^{n+\frac{1}{2}}, \hat{E}^{n+1} + \cancel{\hat{E}^n}\rangle - \langle R^t\hat{E}^n, \cancel{\hat{H}^{n+\frac{1}{2}}} + \hat{H}^{n-\frac{1}{2}}\rangle\Big)$$
$$- \Delta t\Big(\cancel{\langle\omega_p\{\hat{J}\}^{n+\frac{1}{2}}, 2\{\hat{E}\}^{n+\frac{1}{2}}\rangle} - \cancel{\langle\omega_p\{\hat{E}\}^{n+\frac{1}{2}}, 2\{\hat{J}\}^{n+\frac{1}{2}}\rangle}\Big)$$

once again provided $\langle V, b \wedge_h V\rangle = 0$ for all V.

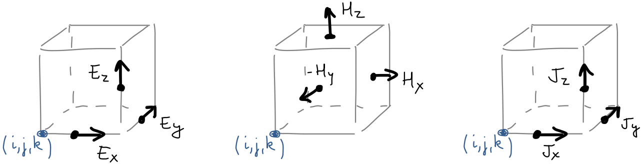
Stability in the energy norm : for $c\Delta t < 2/\|R\| = h/\sqrt{3}$.

Outline

Example of the Xu-Yuan scheme

Based on the Yee scheme for the (E, H) field : general form is

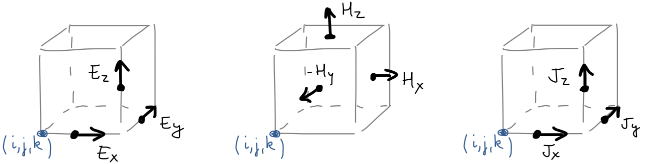
$$\begin{cases} \frac{\varepsilon_0}{\Delta t}(E^{n+1} - E^n) = RH^{n+\frac{1}{2}} - J^{n+\frac{1}{2}} \\ \frac{\mu_0}{\Delta t}(H^{n+\frac{3}{2}} - H^{n+\frac{1}{2}}) = -R^t E^{n+1} \\ \frac{1}{\Delta t}(J^{n+\frac{3}{2}} - J^{n+\frac{1}{2}}) = \varepsilon_0 \omega_p^2 E^{n+1} + \omega_c b \wedge \frac{1}{2}(J^{n+\frac{3}{2}} + J^{n+\frac{1}{2}}). \end{cases}$$



Example of the Xu-Yuan scheme

Based on the Yee scheme for the (E, H) field : general form is

$$\begin{cases} \frac{\varepsilon_0}{\Delta t}(E^{n+1} - E^n) = RH^{n+\frac{1}{2}} - J^{n+\frac{1}{2}} \\ \frac{\mu_0}{\Delta t}(H^{n+\frac{3}{2}} - H^{n+\frac{1}{2}}) = -R^t E^{n+1} \\ \frac{1}{\Delta t}(J^{n+\frac{3}{2}} - J^{n+\frac{1}{2}}) = \varepsilon_0 \omega_p^2 E^{n+1} + \omega_c b\wedge_h \frac{1}{2}(J^{n+\frac{3}{2}} + J^{n+\frac{1}{2}}). \end{cases}$$



Need an explicit solver with the $b\wedge_h$ operator.

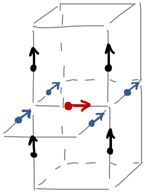
Problem with the X-Y approach

Consider once again the cross product by local averages

$$(b \wedge_h V)_x := b_y\{V_z\} - b_z\{V_y\}$$

$$(b \wedge_h V)_y := \cdots$$

$$(b \wedge_h V)_z := \cdots$$



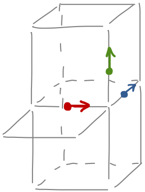
The result

$$\begin{cases} \frac{\varepsilon_0}{\Delta t}(E^{n+1} - E^n) = RH^{n+\frac{1}{2}} - J^{n+\frac{1}{2}} \\ \frac{\mu_0}{\Delta t}(H^{n+\frac{3}{2}} - H^{n+\frac{1}{2}}) = -R^t E^{n+1} \\ \frac{1}{\Delta t}(J^{n+\frac{3}{2}} - J^{n+\frac{1}{2}}) = \varepsilon_0 \omega_p^2 E^{n+1} + \omega_c b \wedge_h \frac{1}{2}(J^{n+\frac{3}{2}} + J^{n+\frac{1}{2}}). \end{cases}$$

is a global scheme which needs a linear solver to invert the matrice.

Solution : use clustered cross-products

Instead, choose a pattern $(\alpha, \beta, \gamma) \in \{-1, +1\}^3$ and define the first order cross product with local clusters :



$$(b \wedge_h V)_x|_{i+\frac{\alpha}{2},j,k} := b_y V_z|_{i,j,k+\frac{\gamma}{2}} - b_z \{V_y\}|_{i,j+\frac{\beta}{2},k}$$

$$(b \wedge_h V)_y|_{i,j+\frac{\beta}{2},k} := b_z V_x|_{i+\frac{\alpha}{2},j,k} - b_x \{V_z\}|_{i,j,k+\frac{\gamma}{2}}$$

$$(b \wedge_h V)_z|_{i,j,k+\frac{\gamma}{2}} := b_x V_y|_{i,j+\frac{\beta}{2},k} - b_y \{V_x\}|_{i+\frac{\alpha}{2},j,k}$$

The resulting scheme

$$\begin{cases} \frac{\varepsilon_0}{\Delta t}(E^{n+1} - E^n) = RH^{n+\frac{1}{2}} - J^{n+\frac{1}{2}} \\ \frac{\mu_0}{\Delta t}(H^{n+\frac{3}{2}} - H^{n+\frac{1}{2}}) = -R^t E^{n+1} \\ \frac{1}{\Delta t}(J^{n+\frac{3}{2}} - J^{n+\frac{1}{2}}) = \varepsilon_0 \omega_p^2 E^{n+1} + \omega_c b \wedge_h \frac{1}{2}(J^{n+\frac{3}{2}} + J^{n+\frac{1}{2}}). \end{cases}$$

can be solved with a local procedure (i.e. solution is explicit and local).

Abstract criterion

The criterion for explicit scheme writes : $(b\wedge_h)^4 = -(b\wedge_h)^2$. Indeed one has the implications

$$J - \alpha b \wedge_h J = Z,$$

$$J - \alpha^2 (b\wedge_h)^2 J = (I + \alpha b\wedge_h) Z,$$

$$(1+\alpha^2)(b\wedge_h)^2 J = (b\wedge_h)^2 (I + \alpha b\wedge_h) Z,$$

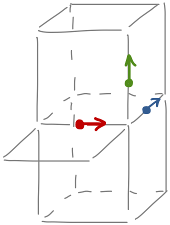
$$J = (I + \alpha b\wedge_h) Z + \frac{\alpha^2}{1+\alpha^2} (b\wedge_h)^2 (I + \alpha b\wedge_h) Z.$$

This algebra is enough to compute the solution by means of explicit and local formulas (for MXYK and new Kernel).

Short summary

The coupling of the Yee scheme and a linear current is

- Stable for : $(V, b \wedge_h V) = 0$
- Explicit for : $(b\wedge_h)^4 = -(b\wedge_h)^2$
- Solution (so far) is clustered <u>first order</u> product

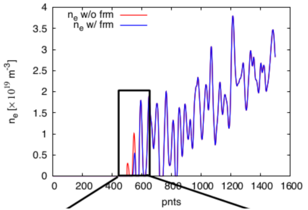


- Additional and natural condition is that ω_p and ω_c are the same within a cluster.

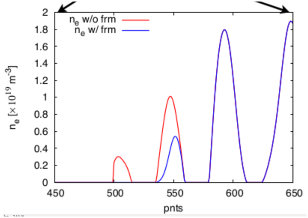
Outline

1. Motivation
2. Stable schemes
3. Explicit schemes
4. Numerical results and perspectives

Data



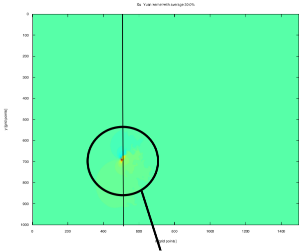
N_e



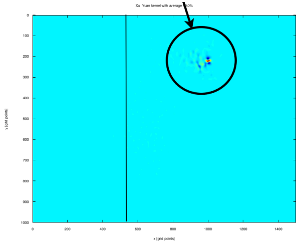
Zoom around the foot of the ramp

Cut of the electronic density in the horizontal direction. An additional kink (in red) is sometimes added at $x = 500$ to evaluate the effect of an extremely strong gradient.

H_z plot



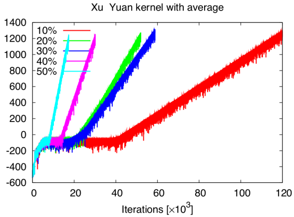
With the kick and 30% noise

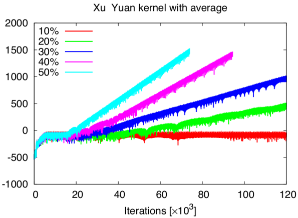


Without the kick but 40% noise

An instability shows up near $x = 500$ cells on the left, near $x = 1000$

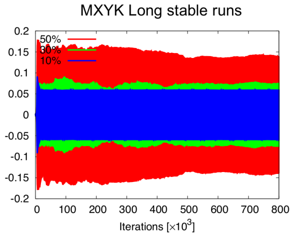
$20 \log_{10} \|H_z\|_{L^\infty}$





- With respect to the time and to the level of noise.
- With the kick on the left, without the kick on the right.

With the first order vectorial product

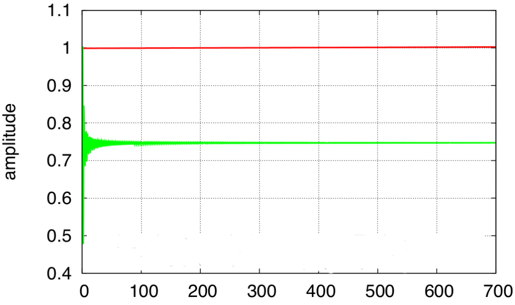


$20 \log_{10} \|H_z\|_{L^\infty}$, with respect to the time and to the level of noise. The computation is done

We observe unconditional stability, with however more amplitude for a higher level of noise. The number of time steps is much greater than in previous figure to illustrate the long time stability of the method.

Energy dissipation

Initial data is a Dirac mass, at the exact foot of the electronic density ramp. The external magnetic field used in this set of runs was $B_0 = 0.95\,T$. The plasma density $N_e(x)$ is linear, with its edge at $x = 500$ grid point. The number of iterations considered is $N = 700$ (far from PML layer).



(slide courtesy of F. Da Silva)

Additional remarks and perspectives

- Need to use clustered multiplications by scalar fields, consistent with clustered cross products.
- Counter-intuitive : the stable and explicit scheme is globally first order (and not second order like the standard Yee scheme).
- Possibility to average in time by alternating the cluster patterns (α, β, γ) in $\{-1, +1\}^3$
- Work in progress for direct simulation of time-dependent densities $N_e = N_e(\mathbf{x}, t)$ (Doppler reflectometry)
- A paper is being written

An open question (is it really ?)

Look at

$$\begin{cases} -\varepsilon_0 \partial_t E + \nabla \wedge H = -q_e N_e(\mathbf{x}) u_e, \\ \mu_0 \partial_t H + \nabla \wedge E = 0, \\ m_e \partial_t u_e = -q_e (E + B_0(\mathbf{x}) \wedge u_e) - \nu m_e u_e \end{cases}$$

plus harmonic forcing on the boundary, plus initial condition, plus friction $\nu > 0$.

Assume resonance configuration (cyclotron, hybrid, ...) : do we have

$$\lim_{\nu \to 0^+} \lim_{T \to \infty} = \lim_{T \to \infty} \lim_{\nu \to 0^+} \quad ?$$

In other words, do we have **Limit absorption=Limit amplitude ?**

If not, which one is the correct physical solution ?

$\lim_{\nu\to 0^+} \lim_{T\to\infty}$ (L.M. Imbert-Grard)