ESF Exploratory Workshop on
MULTI-SCALE METHODS FOR WAVE AND TRANSPORT PROCESSES IN FUSION PLASMAS:
THE LEGACY OF GRIGORY PEREVERZEV

— o —

Wave Theory Agenda

Omar Maj

Division NMPP,
Max Planck Institute for Plasma Physics,
Garching bei München, Germany.

Garhing b. München, 14.10.2013

Brainstorming: a general framework for semiclassical wave equations

Waves in fusion plasma physics

- Wave heating and current drive
- Diagnostics

Kinetic theory of plasma response and wave-particle interactions

Pereverzev's paraxial WKB method

Wave kinetic equation

Summary

Brainstorming: semiclassical wave equations

General structure of high-frequency linear wave equations

- The wave field is described in the frequency domain and generically denoted by

$$u^\varepsilon = u^\varepsilon(\omega, x), \quad 0 < \varepsilon \ll 1, \quad \omega \in \mathbb{R}, \quad x \in \Omega \subseteq \mathbb{R}^d,$$

  where ε and the frequency ω are independent parameters.
- Particularly, ε encodes the scale separation,

$$\varepsilon = \underbrace{\lambda(\omega)}_{\text{wavelength}} \;/\; \underbrace{L}_{\text{plasma scale}} \to 0.$$

- The field is generally governed by a (pseudo-differential) equation

$$p^\varepsilon(\omega, x, -i\varepsilon\nabla)u^\varepsilon(\omega, x) = 0,$$

  defined by the formal substitution $\xi \to -i\varepsilon\nabla$ in a given function $p^\varepsilon(\omega, x, \xi)$.
- The wave field can be either scalar (e.g., the scalar potential ϕ),

$$u^\varepsilon(\omega, x),\ p^\varepsilon(\omega, x, \xi) \in \mathbb{C},$$

  or multi-component (e.g., the electric field E)

$$u^\varepsilon(\omega, x) = \big(u_i^\varepsilon(\omega, x)\big)_i \in \mathbb{C}^N, \quad p^\varepsilon(\omega, x, \xi) = \big(p_{ij}^\varepsilon(\omega, x, \xi)\big)_{ij} \in \mathbb{C}^{N\times N}.$$

- Wave packets: ω-dependence is dropped and $x = (t, r)$ with r the spatial position.

(Remark: The dependence on the frequency ω is implied when not needed.)

- Mappings between phase space functions $p^\varepsilon(x,\xi)$ and operators $p^\varepsilon(x,-i\varepsilon\nabla)$ are referred to as **quantization maps**:

  precise definition of the formal substitution $\xi \to -i\varepsilon\nabla$.
- Infinitely many quantizations exist and (if properly applied) they are all equivalent.
- The Weyl quantization appears to be the most convenient

$$p^\varepsilon(x,-i\varepsilon\nabla)u^\varepsilon(x) = (2\pi\varepsilon)^{-d}\int e^{i(x-x')\cdot\xi/\varepsilon}p^\varepsilon\big(\tfrac{x+x'}{2},\xi\big)u^\varepsilon(x')dx'd\xi.$$

- The function $p^\varepsilon(x,\xi)$ is called **symbol** of the operator.
- Any (semiclassical) partial differential operator is included. Example:

$$\big[\varepsilon^2\Delta + n^2(x)\big]u^\varepsilon(x) = p^\varepsilon(x,-i\varepsilon\nabla)u^\varepsilon(x), \qquad p^\varepsilon(x,\xi) = -\xi^2 + n^2(x).$$

  In the example,

$$p^\varepsilon(x,\xi) = e^{-ix\cdot\xi/\varepsilon}p^\varepsilon(x,-i\varepsilon\nabla)e^{+ix\cdot\xi/\varepsilon},$$

  but it is not always so straightforward!
- In general $p^\varepsilon(x,-i\varepsilon\nabla)$ is a non-local: **Hot plasmas have a non-local response.**

Brainstorming: semiclassical wave equations

Basic hypotheses

- Semiclassical expansion:

$$p^{\varepsilon}(x,\xi) \sim p_0(x,\xi) + \varepsilon p_1(x,\xi) + \varepsilon^2 p_2(x,\xi) + \cdots .$$

- **Assumptions:**
  1. **(WD)** weak dissipation: the leading term (principal symbol) is Hermitian,
  $$p_0^*(x,\xi) = p_0(x,\xi).$$
  2. **(MC)** no linear mode conversion: the real eigenvalues $\lambda_j(x,\xi)$ of $p_0(x,\xi)$ have constant multiplicity and are well separated, namely,
  $$\Big|\lambda_j(x,\xi) - \lambda_k(x,\xi)\Big| \geq C_{jk} > 0, \quad j \neq k.$$
- (WD) is violated for resonant wave-plasma interactions:
  We need to understand wave dynamics in presence of a strongly non-Hermitian operators *(cf. talk by R. Schubert and previous work by E. Westerhof)*.
- If (WD) is fulfilled, then (MC) can be dropped: linear mode conversion theory [Friedland, Kaufman, Tracy, et al.].
- Remark: This formulation of (MC) assumes (WD).

Brainstorming: semiclassical wave equations

Characteristic variety (aka dispersion relation)

- Not every such operator describes wave propagation.
- Hyperbolicity condition:

$$\mathrm{Char}(p^\varepsilon) = \{(x,\xi) \in \mathbb{R}^{2d} : \det p_0(x,\xi) = 0\} \neq \emptyset.$$

- The set $\mathrm{Char}(p^\varepsilon)$ is the characteristic variety and represents geometrically the dispersion relation of the wave.
- Very simple example in one dimension:

$$p^\varepsilon(x,\xi) = -\xi + 1, \quad p^\varepsilon(x, -i\varepsilon\nabla)u^\varepsilon(x) = i\varepsilon \frac{du^\varepsilon(x)}{dx} + u^\varepsilon(x) = 0,$$
$$u^\varepsilon(x) \propto e^{ix/\varepsilon},$$

  the solution is oscillatory with $O(\varepsilon)$ wavelength.
- If the characteristic variety is empty, again in one dimension,

$$p^\varepsilon(x,\xi) = i\xi + 1, \quad p^\varepsilon(x, -i\varepsilon\nabla)u^\varepsilon(x) = \varepsilon \frac{du^\varepsilon(x)}{dx} + u^\varepsilon = 0,$$
$$u^\varepsilon(x) \propto e^{-x/\varepsilon},$$

  the solution is exponential. Those solutions are called evanescent waves.
- Propagating and evanescent waves can appear together: the operator can change signature from hyperbolic to elliptic, depending on the geometry of $\mathrm{Char}(p^\varepsilon)$.

Given the wave equation (restoring the frequency dependence)

$$p^\varepsilon(\omega, x, -i\varepsilon\nabla)u^\varepsilon(\omega, x) = 0,$$

we can pose several physically relevant problems.

1. **Initial value problem** - wave beams launched by an antenna:

   ω is a fixed parameter,
   $u^\varepsilon(\omega, x)|_\Sigma = u_0^\varepsilon(\omega, x)$ given on the antenna plane $\Sigma \hookrightarrow \Omega \subset \mathbb{R}^d$,
   plus conditions on the energy flux and field polarization.

2. **Eigenvalue problem** - stable/unstable modes supported by the plasma (similar to cavity modes in electrodynamics):

   find eigenvalue-eigenfunction pair $\big(\omega, u^\varepsilon(\omega, x)\big)$ on a bounded domain $\Omega \subset \mathbb{R}^d$
   with boundary conditions on $\partial\Omega$.

3. **Solvability problems** (often needed in theory):

   given $v^\varepsilon(\omega, x)$ satisfying some regularity conditions, find $u^\varepsilon(\omega, x)$ such that

$$p^\varepsilon(\omega, x, -i\varepsilon\nabla)u^\varepsilon(\omega, x) = v^\varepsilon(\omega, x).$$

Practical examples follow ...

Brainstorming: semiclassical wave equations

Very rough overview of the WKB method and ray theory

- The WKB method is the backbone of semiclassical methods,

$$u^\varepsilon(x) = a^\varepsilon(x) e^{iS(x)/\varepsilon}, \quad a^\varepsilon(x) \sim a_0(x) + \varepsilon a_1(x) + \varepsilon^2 a_2(x) + \cdots .$$

- Under conditions (WD) and (MC), eigenmodes decouple [Littlejohn and Flynn, Emmrich and Weinstein]. *(E.g., cold plasma modes in the talk by O. Lafitte.)*
- Each eigenvalue $\lambda_i(x,\xi)$ with a non-empty characteristic set $\{\lambda_i(x,\xi) = 0\}$ determines a propagation mode.
- For each mode, the field is polarized in the eigenspace corresponding to λ_i.
- Geometrical optics equations (independent of ε!)

$$\begin{cases} H(x, \nabla S) = 0, & \text{(eikonal equation)}, \\ V(x) \cdot \nabla a_0(x) + \frac{1}{2}\big(\mathrm{div} V(x) + 2i\eta(x)\big) a_0(x) = 0, \end{cases}$$

  where $H(x,\xi) = \lambda_i(x,\xi)$, $V(x) = \nabla_\xi H\big(x, \nabla S(x)\big)$, and $\eta(x)$ accounts for damping, phase shifts and polarization transport.
- Geometrical optics rays are the field lines of $V(x)$ and are traced by

$$\frac{dx}{d\tau} = \nabla_\xi H(x,\xi), \quad \frac{d\xi}{d\tau} = -\nabla_x H(x,\xi), \quad H\big(x(\tau), \xi(\tau)\big) = 0.$$

- An energy balance equation is obtained from the transport equation for $a_0(x)$.

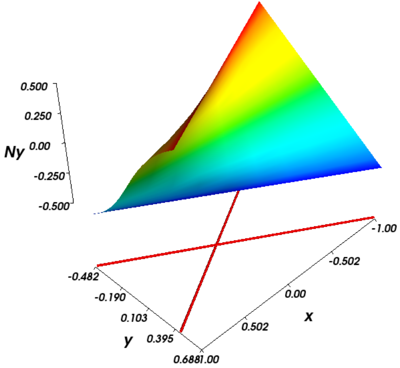
Brainstorming: semiclassical wave equations

Caustics and diffraction effects

In order to construct a solution for the phase $S(x)$ from rays we need

$$\xi(\tau) = \nabla S\big(x(\tau)\big), \quad \text{along rays.}$$

Simple example: focused beam in free space in two dimensions ...



- The congruence of orbits is not everywhere of the form

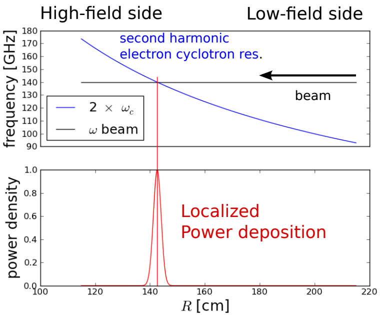
$$\xi = \nabla S(x)!$$

- The WKB ansatz breaks at the focus.
- Description of diffraction effects motivated the development of improved semiclassical methods.
- E.g., **Pereverzev's paraxial WKB**.

1. Heating, current drive and control of magnetized plasmas.
   - Electron cyclotron (EC) frequency:
   $$\omega/2\pi \approx 140\mathrm{GHz}, \quad k_0 = \omega/c \approx 30\mathrm{cm}^{-1}.$$
   - Lower hybrid (LH) frequency:
   $$\omega/2\pi \approx 4 \div 5\mathrm{GHz}, \quad k_0 = \omega/c \approx 1\mathrm{cm}^{-1}.$$
   - Ion cyclotron (IC) frequency:
   $$\omega/2\pi \approx 40\mathrm{MHz}, k_0 = \omega/c \approx 0.008\mathrm{cm}^{-1} \text{ (H}^+\text{ in ASDEX).}$$

   All such schemes rely on the resonant wave-particle interaction.
2. Diagnostics.
   - Reflectometry and Doppler reflectometry (reflection of microwaves from a cut-off). *(Cf. talk by Bruno Després for a numerical method for reflectometry applications).*
   - Electron cyclotron emission (ECE) diagnostic (apply the radiative transfer equation to describe the cyclotron radiation).
3. Linear stability and eigenmodes in plasmas (cf. Transport sessions).

General mechanism for ECRH and ECCD.



- Localized power deposition allows "surgical strikes".
- Selective in the electron phase space due to the resonance condition

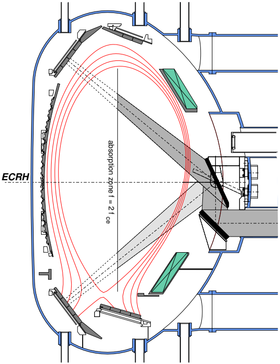
$$\gamma - \omega_c/\omega - N_\parallel \frac{p_\parallel}{m_e c} = 0,$$

where $N_\parallel$ and $p_\parallel$ are the wave refractive index and electron momentum components parallel to the local magnetic field.

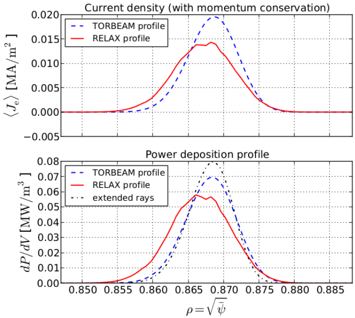
Waves in fusion plasma physics

Wave heating and current drive: ECRH and power deposition profiles

typical set-up (just a sketch)



driven current and power deposition profiles

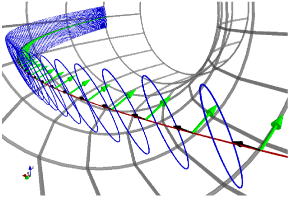


TORBEAM (E. Poli) and RELAX (E. Westerhof)

Typical output:

- Driven electric current averaged over magnetic surfaces.
- Power deposition: density of the absorbed power in the volume shell enclosed by magnetic surfaces.

- Lower hybrid waves are very efficient as a current drive mechanism, e.g., for the control of the q profile:
  - Almost electrostatic wave.
  - Main dissipation mechanism: electron Landau damping.
  - Accessibility and power coupling.
  - Slow group velocity: parametric decay of high-power beams.
  - Spectral gap problem.
  - Full-wave codes are available, but very expensive and require approximate solutions for the interpretation of results.



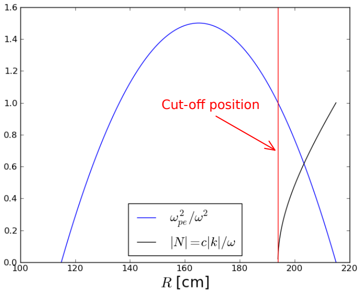
Results of LHBEAM code (N. Bertelli)

- Ion cyclotron waves allow us to heat ion species directly and selectively:
  - The effect of plasma inhomogeneity is stronger.
  - In fusion plasmas, the lower frequency makes semiclassical solutions questionable.
  - Full-wave solvers are available and some are quite fast (TORIC).
  - Quasi-linear effects are crucial for a correct description of the absorption: the iteration between a Maxwell solver and a Fokker-Planck solver is required.

Remark: In the fusion jargon “full-wave solver” means “direct numerical solution of Maxwell’s equations” as opposed to semiclassical asymptotics.

Waves in fusion plasma physics

- Reflectometry concept: “reflection from the edge of the characteristic variety”.
  - The wave beam is reflected from the electron-density-dependent cut-off.
  - The cut-off position is frequency dependent, hence a scan in frequency allows the reconstruction of the electron density profile.
  - The interaction of the beam with turbulent fluctuations near the cut-off gives information on turbulence.
  - Full-wave solvers can be designed, both in frequency and time domain *(cf. talk by B. Després)*.



- Electron cyclotron emission (ECE) measures the temperature of the emitted electron cyclotron radiation.
  - The radiative transfer equation for the transport of the wave spectral intensity is the main modeling tool.

- The “basic” non-relativistic plasma model: Vlasov-Maxwell with Coulomb collisions. Plasma particles of the species α are described by the density

$$f_\alpha(t,x,v)dxdv,$$

measuring the distribution of particles in phase space and satisfying

$$\partial_t f_\alpha + v\cdot\nabla_x f_\alpha + \frac{q_\alpha}{m_\alpha}\Big(E + \frac{v\times B}{c}\Big)\cdot\nabla_v f_\alpha = \sum_\beta C^{\alpha/\beta}(f_\alpha, f_\beta),$$

$$C^{\alpha/\beta}(f_\alpha, f_\beta) = \nabla_v\cdot\Big[D(f_\beta)\nabla_v f_\alpha + A(f_\beta)f_\alpha\Big]$$

+ self-consistent Maxwell’s equations for E and B.

- Quasi-linear theory (heuristic arguments). Stationary solutions of the form

$$f_\alpha(t,x,v) = F_\alpha(x,v) + \mathrm{Re}\Big[e^{-i\omega t}\tilde f_\alpha(\omega,x,v)\Big],$$

$$E(t,x) = \mathrm{Re}\Big[e^{-i\omega t}\tilde E(\omega,x)\Big],$$

$$B(t,x) = B_0(x) + \mathrm{Re}\Big[e^{-i\omega t}\tilde B(\omega,x)\Big].$$

- Average over the fast time scale,

$$\langle f_\alpha(\cdot,x,v)\rangle_\omega = F_\alpha(x,v),\quad \langle E(\cdot,x)\rangle_\omega = 0,\quad \langle B(\cdot,x)\rangle_\omega = B_0(x) = \text{conf. field.}$$

- On making use of the ansatz into the Vlasov-Fokker-Planck equation with collisions linearized over a Maxwellian background and averaging in time, one has

$$v \cdot \nabla_x F_\alpha + (v \times \Omega_\alpha) \cdot \nabla_v F_\alpha = -\nabla_v \cdot \frac{1}{2}\mathrm{Re}\Big[\frac{q_\alpha}{m_\alpha}\Big(\tilde{E}^* + \frac{v \times \tilde{B}^*}{c}\Big)\tilde{f}_\alpha\Big] + C^\alpha(F_\alpha),$$

$$\Omega_\alpha(x) = q_\alpha B_o(x)/m_\alpha c, \quad C^\alpha(F_\alpha) = \nabla_v \cdot \Big[D_c^\alpha(x,v)\nabla_v F_\alpha + A_c^\alpha(x,v)F_\alpha\Big].$$

- The remaining part of the Vlasov equation determines the perturbation

$$-i\omega \tilde{f}_\alpha + v \cdot \nabla_x \tilde{f}_\alpha + (v \times \Omega_\alpha) \cdot \nabla_v \tilde{f}_\alpha = -\frac{q_\alpha}{m_\alpha}\Big(\tilde{E} + \frac{v \times \tilde{B}}{c}\Big) \cdot \nabla_v F_\alpha,$$

  + collisions neglected,
  + nonlinearities neglected (along with nonlinear harmonics generation: $e^{\pm i2\omega}$, $e^{\pm i3\omega}, \ldots$)

- Accepting the approximations, the system has a “certain elegance”:

$$\begin{cases} \mathcal{L}_\alpha F_\alpha = Q^\alpha(\tilde{E}, \tilde{B}, F_\alpha) + C^\alpha(F_\alpha), \\ \text{+ self-consistent Maxwell's equations for for harmonics fields } \tilde{E} \text{ and } \tilde{B}, \end{cases}$$

  where $\mathcal{L}_\alpha = v \cdot \nabla_x + (v \times \Omega_\alpha) \cdot \nabla_v$ describes advection along unperturbed particle orbits.

- At last, one has an advection-diffusion equation in phase space,

$$\mathcal{L}_\alpha F_\alpha = \nabla_v \cdot \Big[D^\alpha \nabla_v F_\alpha + A_c^\alpha F_\alpha \Big],$$

$$D^\alpha = \underbrace{D_c^\alpha(x,v)}_{\text{linearized coll. diffusion}} + \underbrace{D_{ql}^\alpha(x,v,\tilde{E},\tilde{B})}_{\text{quasi-linear diffusion}}.$$

- The distribution function F_α is supposed to relax to an isotropic limit in the remaining fast variables. Example: gyrophase

$$F_\alpha(x,v) = F_\alpha(x,v_\parallel,v_\perp,\phi) = \hat{F}_\alpha(x,v_\parallel,v_\perp) + \text{correction},$$

  and averaging over ϕ,

$$v_\parallel \nabla_\parallel \hat{F}_\alpha - \frac{v_\perp^2}{2B_0}\frac{dB_0}{ds}\left[\frac{\partial \hat{F}_\alpha}{\partial v_\parallel} - \frac{v_\parallel}{v_\perp}\frac{\partial \hat{F}_\alpha}{\partial v_\perp}\right] = \frac{\partial}{\partial v_\parallel}\Big[\Gamma_\parallel^\alpha\Big] + \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\Big[v_\perp \Gamma_\perp^\alpha\Big].$$

- Accounting for the property of the particle orbit (constants of motion and adiabatic invariants) one has better averages, namely,
  - Bounce average.
  - Orbit average.
- Example: RELAX is a bounce-averaged relativistic Fokker-Planck solver for electrons *(cf. the talk by E. Westerhof)*.
- Averaging techniques are at the basis of gyrokinetic theory *(cf. talk by N. Tronko).*

- At last, let us consider the coupled Maxwell's equations

$$\begin{cases} \nabla \times \tilde{E} - i\dfrac{\omega}{c}\tilde{B} = 0, \\ \nabla \times \tilde{B} + i\dfrac{\omega}{c}\tilde{E} = \dfrac{4\pi}{c}\tilde{J}, \end{cases} \qquad \tilde{J}(\omega, x) = \sum_\alpha q_\alpha \int_{\mathbb{R}^d} v\tilde{f}_\alpha(\omega, x, v)dv.$$

- From the linearized Vlasov equation (formally),

$$\begin{aligned} \tilde{f}_\alpha(\omega, x, v) &= \big[-i\omega + \mathcal{L}_\alpha\big]^{-1}\Big[-\frac{q_\alpha}{m_\alpha}\Big(\tilde{E} + \frac{v \times \tilde{B}}{c}\Big)\cdot \nabla_v F_\alpha\Big], \\ &= \big[-i\omega + \mathcal{L}_\alpha\big]^{-1} \circ \mathcal{M}_\alpha \tilde{E}. \end{aligned}$$

- Then $\tilde{J}$ is the result of a non-local operator acting on $\tilde{E}$.
- Second-order equation for the electric field only

$$\nabla \times \big(\nabla \times \tilde{E}\big) - \frac{\omega^2}{c^2}\hat{\epsilon}\tilde{E} = 0, \quad \hat{\epsilon}\tilde{E} = \tilde{E} + \frac{4\pi i}{\omega}\tilde{J} = \text{dielectric operator.}$$

- Semiclassical structure: after normalization $x \to Lx$ we have

$$\kappa = \frac{\omega L}{c} = \frac{1}{\varepsilon}.$$

- **Is $\hat{\epsilon}$ really a pseudo-differential operator?**

- Simplified plasma models for high frequencies.
- Ion at rest: too heavy to respond to the fast wave disturbance.
- Electrons move coherently with velocity $v_e(t, x)$,

$$m_e \partial_t v_e = -e\big(E + v_e \times B/c\big),$$
$$J = -e n_e v_e.$$

  This is called cold plasma model, as electrons have no thermal spread.
- The corresponding dielectric operator reduces to matrix multiplication by

$$\epsilon = \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix}$$

  with S, D, P depending on ω, Ω_e, and ω_{pe} only.
- As far as one is <u>not</u> interested in absorption, this is a good approximation.
- The relevant wave equation reduces to a partial differential equation and can be solved numerically *(cf. the talk by B. Després)*.
- Two propagation modes: O-mode and X-mode *(cf. talk by O. Lafitte)*.
- In the time domain, the cold plasma model amounts to a symmetric hyperbolic system.

Pereverzev's paraxial WKB method

Classical idea of paraxial beams

- Let us split the coordinates according to $x = (y, z) \in \mathbb{R}^{d-1} \times \mathbb{R}$:
  - $y \in \mathbb{R}^{d-1}$ normal directions;
  - $z \in \mathbb{R}$ propagation direction.
- Exact solution of the Helmholtz equation in free space

$$u^\varepsilon(x) = (2\pi\varepsilon)^{-d} \int_{\mathbb{R}^{d-1}} e^{iy\cdot\eta/\varepsilon} e^{iz\sqrt{1-\eta^2}/\varepsilon} \hat{u}^\varepsilon(\eta) d\eta.$$

- When the spectrum $\hat{u}^\varepsilon(\eta)$ is localized near $\eta = 0$, then

$$\sqrt{1-\eta^2} \approx 1 - \frac{1}{2}\eta^2, \quad \text{(paraxial approximation).}$$

- Now a Gaussian spectrum corresponds to a Gaussian beam.

- Idea: build a representation of the transverse structure of the beam, by using an adapted basis instead of Fourier representation.
- Paraxial WKB ansatz

$$u^\varepsilon(x) = A^\varepsilon(x)e^{iS(x)/\varepsilon},$$

$$\begin{aligned} A^\varepsilon(x) &= a(x)\Phi_{mn}\big(v(x)/\sqrt{\varepsilon}\big) \\ &\quad - i\sqrt{\varepsilon}b^\alpha(x)\frac{\partial\Phi_{mn}}{\partial\zeta^\alpha}\big(v/\sqrt{\varepsilon}\big) - \frac{\varepsilon}{2}\frac{\partial^2\Phi_{mn}}{\partial\zeta^\alpha\partial\zeta^\beta}\big(v/\sqrt{\varepsilon}\big) + O(\varepsilon^{3/2}), \end{aligned}$$

- Here, with $d = 3$,

$$\Phi_{mn}(\zeta) = \varphi_m(\zeta^1)\varphi_n(\zeta^2),$$

  and φ_n are the parabolic cylinder functions (Hermite-Gaussian modes).
- The two functions

$$v = (v^1, v^2), \quad v^\alpha = v^\alpha(x), \quad \alpha = 1, 2,$$

  are transverse coordinates to be determined around the curve

$$\mathcal{R} = \{x : v(x) = 0\},$$

  called reference ray. Let $s(x)$ be a third coordinate completing the system.

Pereverzev's paraxial WKB method

Basic equations

- Expansion of pseudo-differential equations **with assumptions (WD) and (MC)**

$$i\varepsilon \frac{\partial p_0}{\partial \xi_i} \frac{\partial A}{\partial x^i} + \frac{\varepsilon^2}{2} \frac{\partial^2 p_0}{\partial \xi_i \partial \xi_j} \frac{\partial^2 A}{\partial x^i \partial x^j} - \Big[p_0 - \frac{i\varepsilon}{2}\Big(\frac{\partial}{\partial x^i}\Big[\frac{\partial p_0}{\partial \xi_i}\Big] + 2ip_1\Big)\Big] A = O(\varepsilon^{3/2}).$$

- Using the ansatz and going through quite a lot of calculations one obtains

$$\begin{cases} H(x, \nabla S) - \frac{1}{2}\Lambda_{\alpha\beta}(s) v^\alpha v^\beta = O(|v|^3), \\ H^\alpha(x, \nabla S, \nabla v) = O(|v|^2) \\ H^{\alpha\beta}(x, \nabla S, \nabla v) - \Delta^{\alpha\beta}(s) = O(|v|), \\ V \cdot \nabla a + \frac{1}{2}\big(\mathrm{div} V + 2i(\eta + \chi_{mn})\big) a = O(|v|), \end{cases}$$

where $H(x, \xi)$ is the considered eigenvalue of p_0,

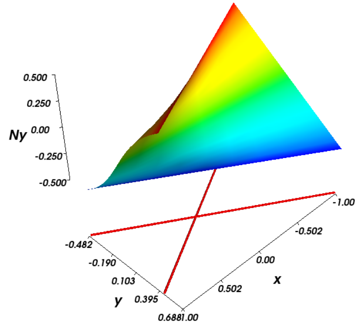
$$H^\alpha = \nabla v^\alpha \cdot \nabla_\xi H(x, \nabla S), \quad H^{\alpha\beta} = \nabla v^\alpha \cdot D^2_\xi H(x, \nabla S) \nabla v^\beta,$$

and $\Lambda_{\alpha\beta}$ and $\Delta^{\alpha\beta}$ are multipliers. This is valid locally near $\mathcal{R} = \{x : v(x) = 0\}$.

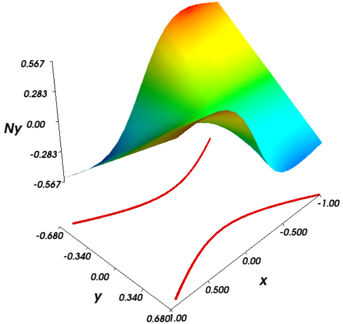
Pereverzev's paraxial WKB method

Deformation of the geometrical-optics Lagrangian manifold near caustics

Geometrical optics



Paraxial WKB



The deformed Lagrangian manifold does not have a critical point.

As a necessary condition, the above system implies the following

- The reference ray $\mathcal{R}$ is a geometrical optics ray.
- Evolution equations along the reference ray,

$$-\frac{dS_{ij}}{d\tau} = \frac{\partial^2 H}{\partial x^i \partial x^j} + S_{ik}\frac{\partial^2 H}{\partial \xi_k \partial x^j} + \frac{\partial^2 H}{\partial x^i \partial \xi_k} S_{kj} + S_{ik}\frac{\partial^2 H}{\partial \xi_k \partial \xi_l} S_{lj} - \phi_{ik}\frac{\partial^2 H}{\partial \xi_k \partial \xi_l}\phi_{lj}$$

$$-\frac{d\phi_{ij}}{d\tau} = \phi_{ik}\frac{\partial^2 H}{\partial \xi_k \partial x^j} + \frac{\partial^2 H}{\partial x^i \partial \xi_k}\phi_{kj} + S_{ik}\frac{\partial^2 H}{\partial \xi_k \partial \xi_l}\phi_{lj} + \phi_{ik}\frac{\partial^2 H}{\partial \xi_k \partial \xi_l} S_{lj}$$

plus a constraint on the initial conditions,

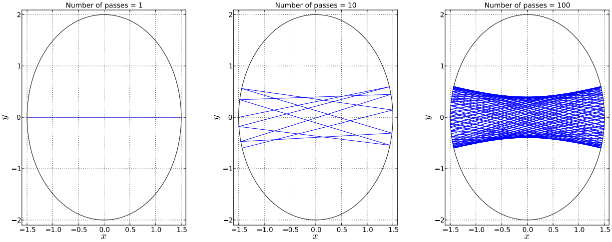
$$\frac{\partial H}{\partial x^i} + S_{ik}\frac{\partial H}{\partial \xi_k} = 0, \quad \phi_{ik}\frac{\partial H}{\partial \xi_k} = 0.$$

- $S_{ij} =$ Hessian matrix of the phase, i.e., phase front curvature.
- $\phi_{ij} =$ quadratic form determining the elliptical beam cross section.
- A relatively small system of ordinary differential equations suffices to reconstruct the wave field.
- This can be proven to be strictly related to complex geometrical optics and extended ray theory *(D. Farina, A. Peeters)*.

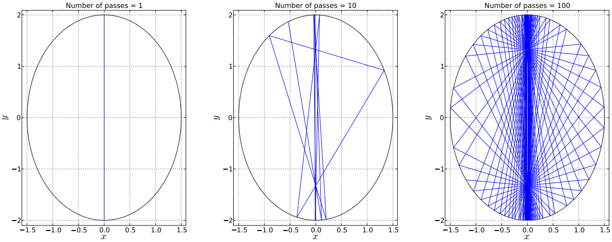
Pereverzev's paraxial WKB method

Ray description of eigenmodes

- Stable orbit



- Unstable orbit



Pereverzev's paraxial WKB method

Relation to the variational Hamilton's equations and stability

- Hamilton's equations (coordinate free form)

$$dz/d\tau = J\nabla H(z),$$

  where J^{-1} is the symplectic form.
- Fixing an orbit $z(\tau)$, we study its stability. Let $z(\tau) + \delta z(\tau)$, then, linearizing,

$$d\delta z(\tau)/d\tau = JD^2 H\big(z(\tau)\big)\delta z(\tau).$$

- The solution of the variational system is

$$\delta z(\tau) = U(\tau)\delta z_0, \quad dU(\tau)/d\tau = JD^2 H\big(z(\tau)\big)U(\tau), \quad U(0) = I,$$

- Upon writing

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \Psi = \big[C + iD\big] \cdot \big[A + iB\big]^{-1}, \quad S = \mathrm{Re}\Psi, \quad \phi = \mathrm{Im}\Psi,$$

  S and Φ solve the paraxial WKB equations. The viceversa also holds true.
- In summary, the paraxial WKB solution entails the information on stability.
- This results establishes a link to Gaussian beams *(cf. the talk by O. Runborg)* ...
- ... as well as to Littlejohn's wave packet method and Maslov's complex WKB.

Wave kinetic equation

Rough summary of the theory

- In random media, the wave field u^ε should be regarded as a random field.
- Wigner matrix

$$W^\varepsilon(x,\xi) = (2\pi\varepsilon)^{-d} \int e^{-i\xi\cdot s/\varepsilon} u^\varepsilon\big(x + \frac{s}{2}\big) u^\varepsilon\big(x - \frac{s}{2}\big)^* ds.$$

- Semiclassical expansion in $\mathcal{S}'(\mathbb{R}^{2d})$,

$$\mathbb{E}(W^\varepsilon) \sim w(x,\xi) e(x,\xi) e^*(x,\xi) + O(\varepsilon), \quad \varepsilon \to 0,$$

  where $\mathbb{E}$ is the expectation value operator.
- The formal limit satisfies

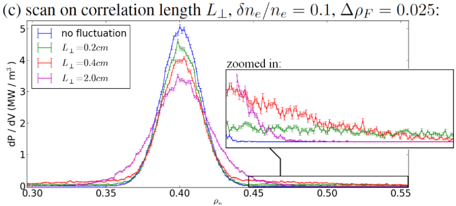
$$\begin{cases} H(x,\xi) w(x,\xi) = 0, \\ \{H, w\}(x,\xi) = -2\gamma(x,\xi) w(x,\xi) + S(w)(x,\xi), \end{cases}$$

  with scattering operator $S(w)$ accounting for the interaction with turbulence.
- A Monte Carlo scheme has been developed for the boundary value problem.
- New wave kinetic code WKBeam at IPP *(H. Weber Master thesis)*.
- Kinetic interpretation of rays *(cf. talk by G. Tanner)*.
- Relationship to the kinetic equations used in turbulence? *(Cf. talk by N. Tronko)*.

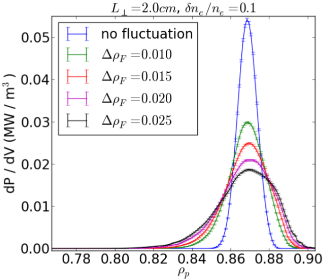
Wave kinetic equation

Preliminary results from WKBeam (Hannes Weber)

- ASDEX-Upgrade



- ITER



Summary

1. Semiclassical methods provide an easy and affordable approach to complicated non-local linear equations.
2. Unfortunately, we are limited to operators with Hermitian principal part: weak dissipation. Non-Hermitian dynamics $\to$ R. Schubert.
3. ECRH and ECCD $\to$ E. Westerhof and D. Farina.
4. LH and IC waves $\to$ A. Cardinali.
5. Quasi-linear theory for the back-reaction on the plasma $\to$ E. Westerhof.
6. Averaging methods in quasi-linear theory.
7. Full-wave solvers for cold plasmas $\to$ B. Després and O. Lafitte.
8. **Pereverzev's paraxial WKB** and beam tracing.
9. Generalizations to phase space and random media?
10. Gaussian beams $\to$ O. Runborg.
11. Eigenvalue problems for linearly unstable eigenmodes.
12. Nonlinear eigenmodes? Which models? Which techniques?
13. Distribution of wave field intensity $\to$ G. Tanner.
14. Wave kinetic equation $\to$ N. Tronko.
15. Beyond fusion: spiral galaxies, solar corona, plasma thrusters $\to$ A. Cardinali.