THE UNIVERSITY *of York*

Exact conservation laws for Gyrokinetic Vlasov-Poisson equations.

Natalia Tronko, York Plasma Institute

Outline

- **Physical motivation**: sources of intrinsic rotation in tokamak
- **Eulerian variational principle** for the full-f and δf GK Vlasov-Poisson system
- **Noether method in fields theory:**
  - Canonical toroidal momentum conservation law derivation for the full-f and δf GK Vlasov-Poisson system
- **Physical interpretation:**
  - From conservation law for canonical quantity towards momentum transport equation for physical quantity

THE UNIVERSITY *of York*

Why consider conservation laws for Vlasov-Poisson system?

- **Gyrokinetic Vlasov-Poisson system**:
  - Consistent description of low-frequency electrostatic micro-turbulence in a tokamak plasma
  - Turbulence $\Longleftrightarrow$ Violent transfer of energy and momentum; growth of instabilities
- ***Conservation laws***:
  - Energy: consistency of dynamical reduction at higher orders, verification accuracy of numerical simulations
  - ***Momentum***: transport phenomena

$\Downarrow$

Intrinsic rotation & plasma stabilization

THE UNIVERSITY *of York*

Momentum transport in tokamak

- **Importance of plasma rotation:**
  - Transition to High confinement H-mode
  - Internal transport barrier formation
- **Sources of plasma rotation**:
  - External torque by neutral beam injection heating system
    - Limitations in future reactor-grade devices
    - Large machine scale, High plasma densities
  - Intrinsic rotation (in H-mode): experimental evidence: JET, Alcator C-Mod and Tore Supra [Rice 2008]

Plasma stabilization & confinement improvement

Small external momentum input

- **Main challenges:**
- Accurately calculate momentum transport equation for reduced models implemented in numerical simulations
- To understand origins of intrinsic rotation

Toroidal angular momentum

- Due to the symmetry of device toroidal rotation is sustained in tokamaks

- **Toroidal angular momentum**: key property for investigation of rotation sources identification

$$\boxed{\sum_{sp} \frac{\partial[[n_{sp} m_{sp} R^2 \Omega]]}{\partial t} + [[\nabla \cdot \Gamma_\phi]] = [[S_\phi]]}$$

- **Goal:** consistently with reduction procedure identify and analyse sources and fluxes

Vlasov momentum approach

- **Standard approach**: momentum transport derived from the GK Vlasov equation
- Analysis of Reynolds stress tensor to identify intrinsic rotation sources

- Non diffusive part

$$\langle v_r \; v_\phi \rangle = \boxed{-\chi_\phi \frac{\partial \langle v_\phi \rangle}{\partial r}} + \boxed{V \langle v_\phi \rangle} + \boxed{\Pi_{r,\phi}}$$

- Diffusive term
  Proportional to toroidal velocity gradient

- Pinch term is
  Proportional to toroidal velocity
  [Peeters PRL2007]

- Residual stress vs Polarization drift **[McDevitt PRL 2008]**
- ϱ_* contributions
- What is its structure?

This work: Establish a unified theoretical framework from variational principle for GK Vlasov-Poisson system

- Momentum conservation law ⟷ momentum transport equation

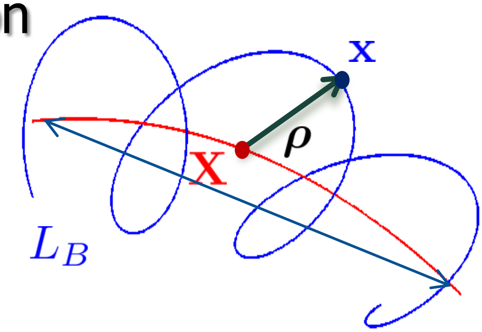
Conservation laws from the variational principles

Variational principles implementation

- *Lagrangian variational principle* **[Scott & Smirnov 2010]**
  - Determination of conserved quantity via Noether's method (particle's level)
  - GK Vlasov momentum calculation for writing canonical angular momentum density conservation law
- *Eulerian variational principle (this work)* **[Brizard & Tronko2011]**
  - Treats particle's via Vlasov distribution as one of the dynamical fields
  - Allows consistent truncation of Lagrangian and corresponding conservation laws derivation

Two-step Gyrokinetic reduction: concept

Charged particle in strong magnetic field: multi-scale motion



Nonuniformity of magnetic field

$$\rho_\perp \left|\nabla_\perp \ln B\right| \ll 1$$

Electromagnetic field fluctuations

$$\epsilon_\delta = \underbrace{(k_\perp \rho_{th})}_{\epsilon_\perp} \frac{e\phi}{T}$$

[Littlejohn 1979,1981]

[Brizard 1989]

- GK ordering $\quad \epsilon_B \sim \epsilon_\delta$

Main goal: systematic elimination of fast scale of motion from dynamical description via near-identity phase space transformations

Modern GK theory: uses language of differential geometry

Two-step Gyrokinetic reduction: particle's space

- **Non-canonical (local) phase space coordinates:** more adapted for dynamical reduction

$$z^\alpha = (\mathbf{X}, p_{||}, \mu, \zeta)$$

- **Generated by the phase-space vector fields**

$$\mathsf{G}_n = \left(G_n^{\mathbf{x}}, G_n^{p_{||}}, G_n^{\mu}, G_n^{\zeta}\right)$$

$$z^\alpha \equiv \tau_\epsilon\, z_0^\alpha = z_0^\alpha + \epsilon G_1^\alpha + \epsilon^2 \left(G_2^\alpha + \frac{1}{2} G_1^\beta \, \frac{\partial G_1^\alpha}{\partial z_0^\beta} \right) + \ldots$$

$(\mathbf{X}, p_{||})$ ■Non-canonical reduced phase space

$\mu = \dfrac{p_\perp^2}{2mB}$ ■Magnetic moment = (slow) variable

ζ ■Gyroangle = Ignorable (fast) variable

Final goal: at each order of coordinate transformation $\dot{\mu} = 0$

- Dynamics in new coordinates: reduction effects
- May be included into Hamiltonian or Symplectic structure

dynamical reduction

Induction of the GK reduction on dynamics

$$\mathcal{T}_\epsilon \equiv \mathcal{T}_{\epsilon_\delta} \circ \mathcal{T}_{\epsilon_B}$$

- **Particle**

$\mathcal{Z}_0$

$(\mathbf{x}, \mu_0, \zeta_0, p_{||0})$

$\mathcal{T}_\epsilon^{-1}$

- **Gyrocenter**

$\mathcal{Z}$

$(\mathbf{X}, \mu, \zeta, p_{||})$

$$F = F_{gy} \circ \tau_\epsilon \equiv \mathsf{T}_{gy} F_{gy}$$

- Pull-back

$$F_{gy} = F \circ \tau_\epsilon^{-1} = \mathsf{T}_{gy}^{-1} F$$

- Push-forward

- Scalar invariance $F(\mathcal{Z}_0) = F_{gy}(\mathcal{Z})$

- Connection with generating fields

$$\mathsf{T}_\epsilon^{\pm} = 1 \pm \pounds_\epsilon$$

$$\pounds_\epsilon \Gamma \equiv \mathsf{G}_n \cdot \mathsf{d}\Gamma + \mathsf{d}\left(i_{G_n}\Gamma\right)$$

Gyrocenter Lagrangian mechanics

- Gyrocenter phase space Lagrangian (1-form) in Hamiltonian representation

$$\Lambda_{gy} \equiv \mathsf{T}_{gc}^{-1}\Gamma_0 - \mathsf{T}_{gy}^{-1}H = \left[\underbrace{\left(\frac{e}{c}\mathbf{A} + p_{||}\hat{\mathsf{b}}\right)}_{\mathbf{p}_{gy}\equiv\mathbf{A}^*}\cdot\mathsf{d}\mathbf{X} + \frac{mc}{e}\mu\mathsf{d}\zeta\right] - H_{gy}\mathsf{d}t$$

- Particle's variational principle

$$0 = \delta\int\Lambda_{gy}$$

- Gyrocenter Euler-Lagrange equations

$$\frac{c}{e}\frac{d_{gy}\mathbf{X}}{dt} \times \mathbf{B}^* - \frac{d_{gy}p_{||}}{dt}\hat{\mathbf{b}} = \nabla H_{gy}$$

$$\hat{\mathbf{b}}\cdot\frac{d_{gy}\mathbf{X}}{dt} = \frac{\partial H_{gy}}{\partial p_{||}} = \frac{p_{||}}{m}$$

Gyrocenter Hamiltonian

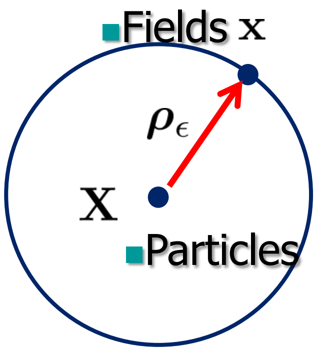
- Hamiltonian representation of reduction procedure:
  - uses the guiding-center Poisson bracket
  - keeps all gyrocenter contributions in the Hamiltonian
- Guiding-center Hamiltonian $H_{gc} = \mathsf{T}_{gc}^{-1} H_0 = \mu B + \frac{p_{||}^2}{2m}$
- Gyrocenter Hamiltonian $H_{gy} = \mathsf{T}_{gy}^{-1} \left(H_{gc} + e\,\epsilon_\delta \phi_{1gc} \right) = \mu B + \frac{p_{||}^2}{2m} + e\,\epsilon_\delta \phi_{1gy}$
- Guiding-center electric potential $\phi_{1gc} = \mathsf{T}_{gc}^{-1} \phi_1 = \phi_1(\mathbf{X} + \boldsymbol{\rho}_{gc}, t) \equiv \widetilde{\phi}_{1gc} + \langle \phi_{1gc} \rangle$
- GK electric potential

$$\begin{aligned} \epsilon_\delta\, \phi_{1gy} \equiv \quad & \epsilon_\delta \left\langle \mathsf{T}_{gy}^{-1} \phi_1 \right\rangle = \epsilon_\delta \phi_1 \left(\mathbf{X} + \boldsymbol{\rho}_\epsilon, t \right) = \\ = \quad & \epsilon_\delta \left\langle \phi_{1gc} \right\rangle - \frac{\epsilon_\delta^2}{2} \left\langle \left\{ S_1, \phi_{1gc} \right\}_{gc} \right\rangle + \cdots \equiv \epsilon_\delta \left\langle \mathsf{T}_{gy}^{-1} \phi_{1gc} \right\rangle \end{aligned}$$

- Gyrocenter generating function

$$\boxed{S_1 \equiv \frac{e}{\Omega} \int \widetilde{\phi}_{1gc} d\zeta}$$

Guiding-center & Gyrocenter Polarization



- **Dynamical reduction:**

- Fields and particles are not evaluated at the same position

- Gyrocenter position:

$$\mathbf{X} = \mathsf{T}_{\epsilon_\delta}^{-1}\left[\mathsf{T}_{\epsilon_B}^{-1}\mathbf{x}\right] \quad = \quad \mathbf{x} - \boldsymbol{\rho}_0 - \epsilon\boldsymbol{\rho}_1 - \epsilon^2\boldsymbol{\rho}_2 + \dots \equiv$$

$$\mathbf{x} - \underbrace{\underbrace{\left(\boldsymbol{\rho}_{00} + \epsilon_B\boldsymbol{\rho}_{01} + \dots\right)}_{\boldsymbol{\rho}_{gc}} + \epsilon_\delta\left(\boldsymbol{\rho}_{10} + \epsilon_B\boldsymbol{\rho}_{11} + \dots\right)}_{\boldsymbol{\rho}_\epsilon}$$

- Gyrocenter displacement

- Gyrokinetic delta-function: brings fields and particles at the same position

$$\frac{\delta\phi_{1gy}(\mathbf{X})}{\delta\phi_1(\mathbf{x})} = e\,\epsilon\left\langle \mathsf{T}_{gy}^{-1}\left[\delta^3(\mathbf{X} + \boldsymbol{\rho}_{gc} - \mathbf{x})\right]\right\rangle = e\,\left\langle \mathsf{T}_{gy}^{-1}\delta_{gc}^3\right\rangle \equiv e\left\langle \delta^3\left(\mathbf{X} + \boldsymbol{\rho}_\epsilon - \mathbf{x}\right)\right\rangle$$

Eulerian Variational principle for GK Vlasov-Poisson system

- **GK Vlasov-Poisson action:** coupling of reduced particle dynamics with electromagnetic fields

$$\mathcal{A}_{gy} = \boxed{-\int d^8 Z\ \mathcal{F}_{gy}(\mathcal{Z})\,\mathcal{H}_{gy}(\mathcal{Z},\phi_{1gy})} + \int \frac{d^4x}{4\pi}\left(\epsilon^2|\mathbf{E}_1|^2 - |\mathbf{B}_0|^2\right)$$

- gy:Gyrocenter dynamics: polarization effects

[Brizard 2000]

$$d^8 Z \equiv c^{-1} d\mathcal{E}\, dt\, d\mathbf{X}\, dp_{||} d\mu \qquad d^4x \equiv c^{-1}\, dt\, d\mathbf{X}$$

- Extended 8D phase space

- Hamiltonian dynamics on extended phase space satisfies physical constraint $\mathcal{H} = 0 \rightarrow H = \mathcal{E}$

$$\{\mathcal{E}\,,\,t\} = 1$$

$$(\mathcal{E}, t;\ \mathbf{X}, p_{||}, \mu)$$

- Extended dynamical fields

- Hamitlonian $\mathcal{H}_{gy} = H_{gy} - \mathcal{E}$
- Vlasov distribution $\mathcal{F}_{gy} = c\,\delta\left(\mathcal{E} - H_{gy}\right)\ F\left(\mathbf{X}, p_{||}, \mu;\ t\right)$

Full GK variational principle

- First principle of dynamics

$$0 = \delta \mathcal{A}_{gy} = \int d^4x \, \delta \mathcal{L}_{gy}$$

- Lagrangian density variation

$$\delta \mathcal{L}_{gy} = \frac{\epsilon_\delta^2}{4\pi} \, \delta \, \mathbf{E}_1 \cdot \mathbf{E}_1 - \int d^4p \, \left(\delta \mathcal{F} \, \mathcal{H}_{gy} + \mathcal{F} \, \delta \mathcal{H}_{gy} \right)$$

- Eulerian fields variations

1.Electric field

$$\delta \mathbf{E}_1 = -\nabla \delta \phi_1$$

2.Extended Vlasov distribution

$$\delta \mathcal{F} \equiv \{ \delta \mathcal{S}, \mathcal{F} \}_{extgc}$$

3.Extended Hamiltonian

- Functional derivative: key property for polarization effects identification

$$\delta \mathcal{H}_{gy} = \delta \phi_1 \frac{\delta \mathcal{H}_{gy} \left(\mathbf{X} \right)}{\delta \phi_1 \left(\mathbf{x} \right)} = \delta \phi_1 \frac{\delta \phi_{gy} \left(\mathbf{X} \right)}{\delta \phi_1 \left(\mathbf{x} \right)} = e \left\langle \delta^3 \left(\mathbf{X} + \boldsymbol{\rho}_\epsilon - \mathbf{x} \right) \right\rangle$$

Lagrangian variation

- Mixed variations:

$$\delta\mathcal{L}_{gy} = \delta\mathbf{x}\cdot\left(\nabla\mathcal{L}_{gy} - \nabla'\mathcal{L}_{gy}\right)$$

Contributions of non-dynamical fields

$$\mathbf{B} = \mathbf{B}_0$$

$$F = F_0 + \epsilon F_1$$

∇' Explicit grad with restriction on holding constant dynamical fields

- Lagrangian density variation

$$\begin{aligned}\delta\mathcal{L}_{gy} &= -\epsilon_\delta\,\delta\phi_1\left[\frac{\epsilon_\delta}{4\pi}\nabla^2\phi_1 + e\int\left\langle \mathsf{T}_{gy}^{-1}\delta_{gc}^3\right\rangle F\,d^6Z\right]\\ &- \int \delta S\,\{\mathcal{F},\mathcal{H}\}_{extgc}\,d^4p\\ &+ \frac{\partial\Lambda_{gy}}{\partial t} + \nabla\cdot\mathbf{\Gamma}_{gy}\end{aligned}$$

- Gyrokinetic Noether densities

$$\begin{aligned}\Lambda_{gy} &\equiv \int \delta S\,\mathcal{F}\,d^4p\\ \Gamma_{gy} &\equiv \int \frac{d_{gy}\mathbf{X}}{dt}\,\delta S\,\mathcal{F}\,d^4p + \epsilon_\delta\,\frac{\delta\phi_1\nabla\phi_1}{4\pi}\end{aligned}$$

GK Vlasov-Poisson equations

- GK Vlasov equation

$$
\begin{aligned}
0 &= \{\mathcal{F}, \mathcal{H}\}_{ext\,gc} \equiv \frac{\partial F}{\partial t} + \{F, H_{gy}\}_{gc} \\
&= \frac{\partial F}{\partial t} + \frac{d_{gy}\mathbf{X}}{dt} \cdot \nabla F + \frac{d_{gy}p_{||}}{dt} \frac{\partial F}{\partial p_{||}}
\end{aligned}
$$

- GK Poisson equation: dynamics and quasineutrality condition

$$
\begin{aligned}
\epsilon_\delta \nabla \cdot \mathbf{E}_1 &= -4\pi\, e \int d^3p\; F\, \frac{\delta \phi_{1gy}}{\delta \phi_1} = -4\pi\, e \int d^3p\, F \left\langle \mathsf{T}_{gy}^{-1} \delta_{gc}^3 \right\rangle \\
&\equiv 4\pi \left(\rho_{gy} - \nabla \cdot \mathcal{P}_{gy} \right)
\end{aligned}
$$

- Gyrocenter charge

$$
\varrho_{gy} \equiv \sum_{sp} e \int F\, d^3p
$$

- Gyrocenter polarization

$$
\mathcal{P}_{gy} \equiv \sum_{sp} e \int F\, \langle \boldsymbol{\rho}_{gy} \rangle\, d^3p - \nabla \cdot Q
$$

Eulerian Variational principle for δf-GK Vlasov-Poisson system

- Truncated Vlasov-Poisson action:

direct coupling of reduced particle dynamics with electromagnetic fields

$$\mathcal{A}_{gy}^{(1)} = -\int d^8Z\,\mathcal{F}_{gy}^{(1)}(\mathcal{Z})\,\mathcal{H}_{gy}^{(1)}(\mathcal{Z},\phi_{1gy}) + \frac{\epsilon_\delta^2}{2}\,e\int d^7Z\,F_0\left\langle \{S_1,\phi_{1gc}\}_{gc}\right\rangle + \int \frac{d^4x}{4\pi}\left(\epsilon_\delta^2|\nabla\phi_1|^2 - |\mathbf{B}_0|^2\right)$$

[Brizard 2010]

$$d^8Z \equiv c^{-1}d\mathcal{E}\,dt\,d\mathbf{X}\,dp_{||}d\mu$$

$$d^7Z \equiv c^{-1}\,dt\,d\mathbf{X}\,dp_{||}d\mu \qquad d^4x \equiv c^{-1}\,dt\,d\mathbf{X}$$

- Extended dynamical fields

- Hamiltonian

$$\mathcal{H}_{gy}^{(1)} = \left(\frac{p_{||}^2}{2m} + \mu\,B + \epsilon_\delta\,e\,\langle\phi_{1gc}\rangle\right) - \mathcal{E} \equiv H_{gy}^{(1)} - \mathcal{E}$$

- Vlasov distribution

$$\mathcal{F}_{gy}^{(1)} = c\,\delta\left(\mathcal{E} - H_{gy}^{(1)}\right)\,(F_0 + \epsilon_\delta F_1)$$

δf GK Vlasov-Poisson equation

- Truncated δf Vlasov equation

$$\frac{d_{gc}F_1}{dt} = -\{F_0, e\,\langle\phi_{1gc}\rangle\}_{gc} - \epsilon_\delta\,\{F_1, e\,\langle\phi_{1gc}\rangle\}_{gc}$$

- Truncated δf Poisson equation

$$\begin{aligned} \epsilon_\delta\,\nabla\cdot\mathbf{E}_1 &= -4\pi\int e\,\left((F_0+\epsilon_\delta F_1)\,\langle\delta^3_{gc}\rangle - \epsilon_\delta\,F_0\,\left\langle\left\{S_1,\delta^3_{gc}\right\}\right\rangle\right)d^3p\,d^3x \\ &= 4\pi\,(\varrho - \nabla\cdot\mathcal{P}) \end{aligned}$$

- Separation of guiding-center and gyrocenter contributions into the quasineutrality condition

$$\boxed{\varrho = \nabla\cdot\mathcal{P}_{gc} + \epsilon_\delta\;\nabla\cdot\mathcal{P}_{1\;gy}}$$

Noether's method in field theory

- Continuous Lagrangian symmetries

$\Leftrightarrow$

- Conservation laws
- Time $\longleftrightarrow$ Energy
- Space $\longleftrightarrow$ Momentum

- [Brizard 2010]
- [Brizard &Tronko 2011]
→ δf version in preparation

$$\delta\,\mathcal{L}_{gy} \equiv \frac{\partial \Lambda_{gy}}{\partial t} + \nabla\cdot\mathbf{\Gamma}_{gy}$$

- Virtual translations in space

$$\begin{aligned}
\delta\,S &= \mathbf{p}_{gy}\cdot\delta\mathbf{x} \\
\delta\,\phi_1 &= -\delta\mathbf{x}\cdot\nabla\phi_1 \\
\delta\,\mathcal{L}_{gy} &= -\delta\mathbf{x}\cdot\left(\nabla\mathcal{L}_{gy} - \nabla'\mathcal{L}_{gy}\right)
\end{aligned}$$

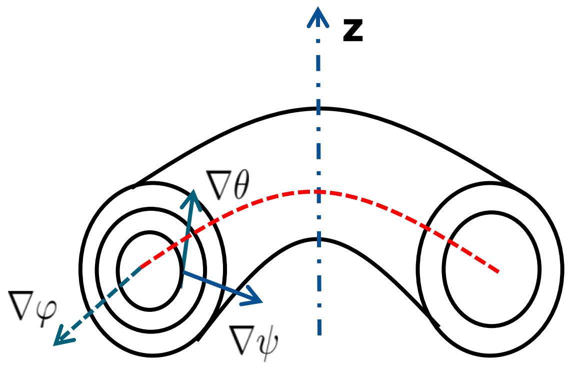
GK canonical angular-momentum conservation law

- Axisymmetric tokamak geometry:

Use symmetry of magnetic field:

$$\delta x \equiv \frac{\partial \mathbf{X}}{\partial \varphi}\delta\varphi$$

$$\mathbf{B} = \nabla\varphi \times \nabla\psi + q(\psi)\nabla\psi \times \nabla\theta$$



- Conservation law for canonical toroidal angular-momentum

$$p_{gy\varphi} \equiv \frac{\partial \mathbf{X}}{\partial \varphi}\cdot \mathbf{p}_{gy} \equiv -\frac{1}{c}\psi + p_{||}\, b_\varphi$$

$$\frac{\partial P_\varphi}{\partial t} = -\nabla\cdot \boldsymbol{R}_\varphi - \int F\left(\frac{\partial H_{gy}}{\partial \varphi}\right) d^3p$$

- Density $\quad P_\varphi = \sum \int F\, p_{gy\varphi}\, d^3p$

- Flux $\quad R_\varphi = \int F \frac{d\mathbf{X}_{gy}}{dt}\, p_{gy\,\varphi}\, d^3p$

- Next step: Separating polarization and kinetic contributions

Main steps towards transport equation

$$\frac{\partial}{\partial t}\left([[P_{||,\varphi}]] + \boxed{\frac{1}{c}\,[[\mathcal{P}^{\psi}]]} \right) + \frac{1}{\mathcal{V}}\frac{\partial}{\partial \psi}\left(\mathcal{V}\left[\left[R^{\psi}_{||,\varphi}\right]\right] \right) = -\epsilon_{\delta} \sum e \left[\left[\int F\,\frac{\partial H_{gy}}{\partial \varphi}\, d^3 p\right]\right]$$

- 1. ***Flux surfaces averaging:*** $[[\ldots]] \equiv \frac{1}{\mathcal{V}} \oint (\ldots)\, \mathcal{J} d\theta\, d\varphi$
- 2. ***Physical constraints:***
  quasineutrality $\varrho = \nabla \cdot \mathcal{P}$
  charge conservation $\partial_t \varrho + \boldsymbol{\nabla} \cdot \mathbf{J} = 0$

- Can we include sources terms as flux contributions?

- What is the physical meaning of the transported quantity?

Full-f polarization effects: *source terms*

- **Source term:** polarization effects, multipole decomposition

$$\frac{\partial H_{gy}}{\partial \varphi} = e\,\epsilon_\delta \frac{\partial \phi_{1gy}}{\partial \varphi} = e\,\epsilon_\delta \left(\frac{\partial \phi}{\partial \varphi} \quad + \quad \langle \boldsymbol{\rho}_\epsilon \rangle \cdot \nabla \frac{\partial \phi_1}{\partial \varphi} \right.$$
$$\left. + \quad \frac{1}{2} \langle \boldsymbol{\rho}_\epsilon \boldsymbol{\rho}_\epsilon \rangle : \nabla\nabla \frac{\partial \phi_1}{\partial \varphi} + \ldots \right)$$

- Leibnitz rule

$$\frac{\partial \phi_1}{\partial \varphi} \left(\nabla \cdot \mathcal{P} \right) = \nabla \cdot \left(\mathcal{P} \frac{\partial \phi_1}{\partial \varphi} \right) - \mathcal{P} \cdot \nabla \frac{\partial \phi_1}{\partial \varphi}$$

- Quasineutrality

$$\int F \frac{\partial H_{gy}}{\partial \varphi} d^3 p \quad = \quad \epsilon_\delta \frac{\partial \phi_1}{\partial \varphi} \boxed{\left(\varrho_{gy} - \nabla \cdot \mathcal{P}_{gy} \right)} \rightarrow 0$$
$$+ \quad \boxed{\nabla \cdot \left(\epsilon_\delta \mathcal{P}_{gy} \frac{\partial \phi_1}{\partial \varphi} + \epsilon_\delta \mathsf{Q}_{gy} \cdot \nabla \frac{\partial \phi_1}{\partial \varphi} \right)}$$

Key property to eliminate source terms

- Fluxes only

δ-f polarization effects: *source terms*

- First order gyrocenter Hamiltonian

$$H_{gy}^{(1)} = \left\langle \mathsf{T}_{gc}^{-1} \left(H_0 + \epsilon_\delta \phi_1 \right) \right\rangle = H_{gc} + \epsilon_\delta \left\langle \phi_1 \left(\mathbf{X} + \boldsymbol{\rho}_{gc} \right) \right\rangle$$

- Multipole decomposition (guiding-center displacements)

$$\frac{\partial H_{gy}^{(1)}}{\partial \varphi} = e\,\epsilon_\delta \; (\; \frac{\partial \phi}{\partial \varphi} \; + \; \langle \boldsymbol{\rho}_{gc} \rangle \cdot \nabla \frac{\partial \phi_1}{\partial \varphi} \; + \; \frac{1}{2} \langle \boldsymbol{\rho}_{gc} \boldsymbol{\rho}_{gc} \rangle : \nabla\nabla \frac{\partial \phi_1}{\partial \varphi} + \ldots)$$

Only partial compensation of source terms from the quasineutrality condition

$$\varrho - \nabla \cdot \mathcal{P}_{gc} = -\epsilon_\delta \; \nabla \cdot \mathcal{P}_{1\;gy}$$

$$\int F \frac{\partial H_{gy}^{(1)}}{\partial \varphi} d^3 p \; = \; e\epsilon_\delta \frac{\partial \phi_1}{\partial \varphi} \boxed{(\varrho - \nabla \cdot \mathcal{P}_{gc})} \;\Rightarrow\; \boxed{-\epsilon_\delta \; \nabla \cdot \mathcal{P}_{1gy}}$$
$$+ \; \nabla \cdot \left(\epsilon_\delta \, \mathcal{P}_{gc} \frac{\partial \phi_1}{\partial \varphi} + \epsilon_\delta \, \mathsf{Q}_{gc} \cdot \nabla \frac{\partial \phi}{\partial \varphi} \right)$$

- Source terms of higher orders

- Consistency: possibility to avoid source terms at the first order

$$\left[\left[\mathcal{P}_{1gy} \cdot \nabla \frac{\partial \phi_1}{\partial \varphi} \right]\right] = - \left[\left[\frac{\partial}{\partial \varphi} \left(\frac{mc^2}{2B^2} \left| \nabla_\perp \phi_1 \right|^2 \right) \right]\right]$$

Polarization: direct definition

$$\mathcal{P}_{gy} \equiv \sum_{sp} e \int F \left\langle \boldsymbol{\rho}_\epsilon \right\rangle d^3p - \nabla \cdot Q$$

$$\boldsymbol{\rho}_\epsilon = \boldsymbol{\rho}_{gc} + \epsilon_\delta \boldsymbol{\rho}_{1gy} + \dots$$

- **Direct (formal) definition:**

  from the near-identity phase space transformation

$$\begin{aligned} \boldsymbol{\rho}_\epsilon &\equiv \mathsf{T}_\epsilon^{-1}\mathbf{x} - \mathbf{X} \\ &= -\epsilon\, G_1^{\mathbf{x}} - \epsilon^2 \left(G_2^{\mathbf{x}} - \frac{1}{2}\mathsf{G}_1 \cdot \mathsf{d}G_1^{\mathbf{x}} \right) + \dots \end{aligned}$$

- **The first order displacement**

  containing guiding-center and gyrcenter contriutions

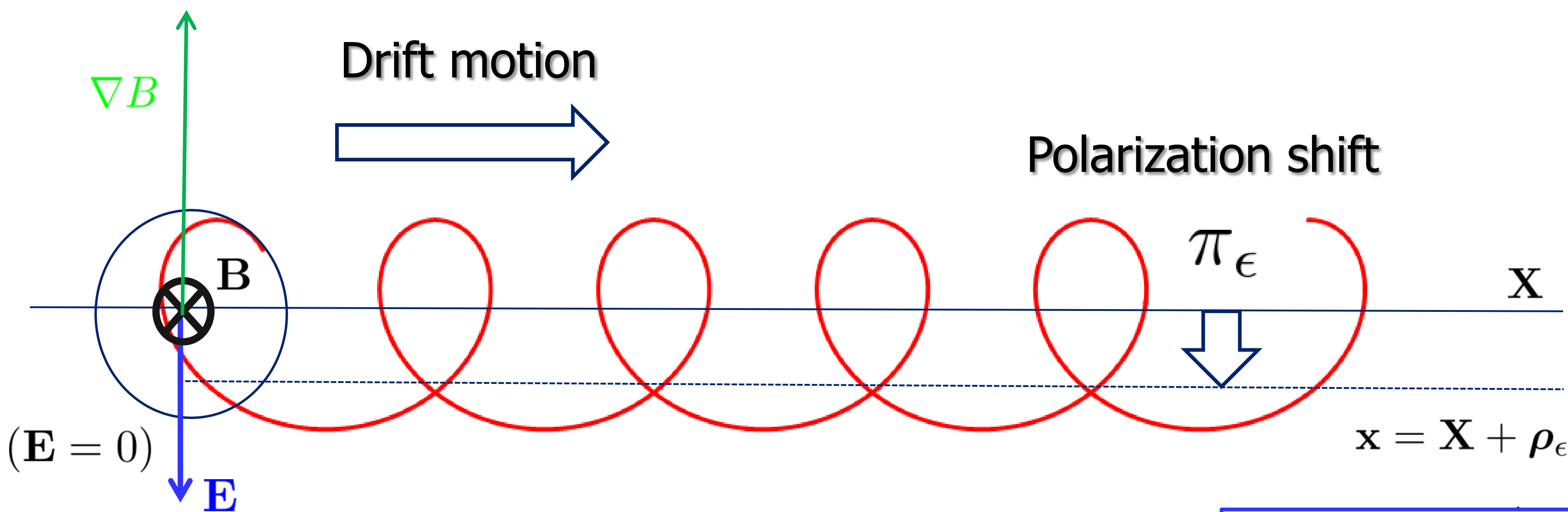
$$\begin{aligned} \left\langle \boldsymbol{\rho}_\epsilon^{(1)} \right\rangle &= \epsilon_B \left\langle \boldsymbol{\rho}_{gc} \right\rangle + \epsilon_\delta \left\langle \boldsymbol{\rho}_{1gy} \right\rangle \\ &= -\frac{1}{m\Omega^2} \left(\mu \nabla B + \frac{p_{||}^2}{m} \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}} \right) - \frac{c}{B\Omega} \nabla_\perp \left\langle \phi_{1gc} \right\rangle \end{aligned}$$

Polarization: physical definition

- Physical (dynamical) definition
  [Pfirsch 1984 & Kaufman 1986]

$$\pi_\epsilon^{(1)} = \left\langle \boldsymbol{\rho}_\epsilon^{(1)} \right\rangle = \frac{\hat{\mathbf{b}}}{\Omega} \times \frac{d_{gy}^{(1)}\mathbf{X}}{dt}$$

$$\left(\frac{d_{gy}^{(1)}\mathbf{X}}{dt} \right)_\perp = \frac{c\hat{\mathbf{b}}}{eB} \times \left[\epsilon_B \left(\mu \nabla B + \frac{p_{||}^2}{m} \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}} \right) + \epsilon_\delta\, e\, \langle \phi_{gc} \rangle \right]$$



$$\mathbf{U}_{\nabla B} = \frac{c\hat{\mathbf{b}}}{eB} \times \mu \nabla B$$

$$\pi_{\nabla B} = \frac{e}{\Omega} \hat{\mathbf{b}} \times \mathbf{U}_{\nabla B}$$

$$\mathbf{U}_\mathrm{E} = \mathbf{E} \times \frac{c\mathbf{b}}{B}$$

$$\pi_\mathrm{E} = \frac{e}{\Omega} \hat{\mathbf{b}} \times \mathbf{U}_\mathrm{E}$$

Recovering the toroidal angular momentum

- Polarization density from physical definition

$$\mathcal{P} = \sum m \frac{c\hat{\mathbf{b}}}{B} \times \left[\int F \left(\frac{d_{gy}^{(1)}\mathbf{X}}{dt} \right)_{\perp} d^3p \right] - \nabla \cdot \mathsf{Q}$$

- Direct geometrical projection

$$\nabla\psi \equiv \mathbf{B} \times \frac{\partial \mathbf{X}}{\partial \varphi}$$

$$\mathcal{P}^{\psi} = \mathcal{P} \cdot \nabla\psi = \sum \frac{mc}{B}\hat{\mathbf{b}} \times \left[\int F \frac{\partial \mathbf{X}}{\partial \varphi} \cdot \frac{d_{gy}^{(1)}\mathbf{X}}{\partial t} d^3p \right] - \sum \left(\int F\, p_{||} d^3p \right) b_{\varphi} - \nabla\psi \left(\nabla \cdot \mathsf{Q} \right)$$

$$\Downarrow$$

$$\mathsf{L}_{\varphi}^{(1)} = m\, \mathcal{R}^2 \frac{d_{gy}^{(1)}\varphi}{dt}$$

$$\Downarrow$$

$$P_{||,\varphi}$$

Physical Interpretation: transport equation

THE UNIVERSITY *of York*

- 1. *Flux surfaces averaging:* $[[\ldots]] \equiv \frac{1}{\mathcal{V}} \oint (\ldots)\, \mathcal{J}\, d\theta\, d\varphi$
- 2. *Physical constraints:*
  quasineutrality $\varrho = \nabla \cdot \mathcal{P}$
  charge conservation $\partial_t \varrho + \boldsymbol{\nabla} \cdot \mathbf{J} = 0$

- 3. *Full gyrocenter polarization displacement : necessary for recovery of physical meaning of transported quantity*

$$\langle \boldsymbol{\rho}_\epsilon \rangle = -\frac{1}{m\Omega^2}\left(\mu \nabla_\perp B + \frac{p_{||}^2}{m}\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}\right) - \frac{c}{B\Omega}\nabla_\perp \langle \phi_{1gc} \rangle$$

$$\frac{\partial}{\partial t}\left([[P_{||\varphi}]] + \frac{1}{c}\,[[\mathcal{P}^\psi]]\right) + \frac{1}{\mathcal{V}}\frac{\partial}{\partial \psi}\left(\mathcal{V}\,\left[\left[R^\psi_{||\varphi}\right]\right] + \epsilon\left[\left[\mathcal{P}^\psi \frac{\partial \phi_1}{\partial \varphi} + \mathbf{Q}^\psi \cdot \nabla \frac{\partial \phi_1}{\partial \varphi} + \cdots\right]\right]\right) = 0$$

- Transferred physical quantity: contains radial polarization
  - No gc contributions
  - [Scott&Smirnov2010]

$$\mathsf{L}^{(1)}_\varphi = m\, \mathcal{R}^2 \frac{d^{(1)}_{gy}\varphi}{dt}$$

- Reynolds tensor $R^\psi_{||\varphi} \sim v_\psi\ v_\varphi$

  standard contributions

- Residual stress tensor: field-particle momentum exchange;
- Turbulent, electric-shear contributions
  - [McDevitt2009]

Conclusions

- Main outcomes of variational formulation :
  - Exactly-conserved quantities (axisymmetric magnetic geometry) even for gk reduced system (δf case included)
  - Possibility to recover a posteriori the variational results from GK Vlasov moments equation
  - No need to use closure for fluid equations
  - Recovery of physical meaning of transported quantity only if all polarization corrections are consistently taken into account consistently with δf truncation
  - Access to the residual stress structure: inertial source of toroidal momentum transport, field-particle interactions
  - Perspectives:
    - considering electromagnetic turbulence case
    - further numerical implementation/ comparison with experiment