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Weak turbulence in two-dimensional Magnetohydrodynamics

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Outline

- **Motivations:** analytical methods for fluid models in plasma physics
- **Turbulence:** HD and MHD in 2D&3D: what is common and what is different?
- **Wave-kinetic** approach for 2D MHD model
- **Dynamics** of spectrum
- **Beyond WT** approach: qualitative picture

Fluid models for magnetized plasmas

Mesoscopic dynamical description

Uses calculation of the Vlasov distribution moments

$$n = \int d^3v \;\; f(\mathbf{x}, \mathbf{v})$$

$$\mathbf{u} = \int d^3v \; \mathbf{v} \; f(\mathbf{v}, \mathbf{x})$$

Requires closure: external condition, which allows to truncate the number of moments necessary for model consistency (incompressibility)

The incompressible MHD

$$\left.\begin{aligned}
\left(\partial_t + \mathbf{u}\cdot\nabla\right)\mathbf{u} &= \frac{\mathbf{B}\cdot\nabla\mathbf{B}}{\mu_0} - \nabla\left(\frac{B^2}{2\mu_0}\right) - \nabla p \\
\frac{\partial \mathbf{B}}{\partial t} &= -\nabla\times\left(\mathbf{u}\times\mathbf{B}\right) \\
\nabla\cdot\mathbf{B} = \nabla\cdot\mathbf{u} &= 0
\end{aligned}\right\}$$

Methods delivering analytical results:

- Wave-turbulence approach:
  - Kinetic equation for spectrum
- Hamiltonian approach:
  - Identification of exact invariants and systematic study of equilibrium stability

Interpretation and verification of numerical simulations

- Validity of 2D MHD simulations
- Explication of new equilibrium states identified numerically

Why studying incompressible 2D MHD model ?

Utility of the model:

•Very simple model: capturing basic mechanism of turbulence

•Development of new numerical methods: better resolution then in 3D

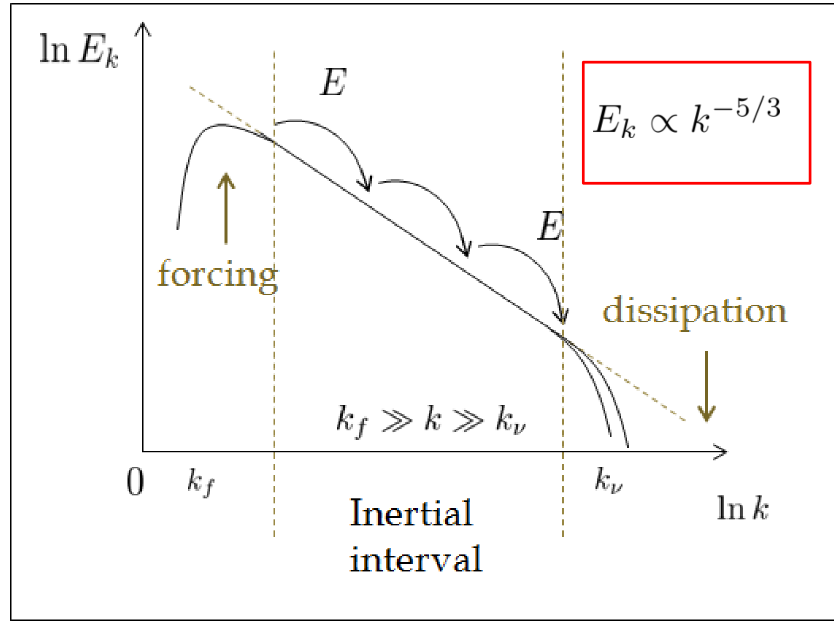
•Astrophysical and fusion plasmas purposes

Fundamental question:

• Do the 2D & 3D MHD turbulence exhibit similar behaviour?

Turbulence: Hydrodynamic and waves

Kolmogorov 1941:
Hydrodynamic turbulence
[Isotropic in 3D]



1. ***Local energy cascade***: energy flux throughout the scales is generated by sequence of transfer between eddies of the same size
    - Direct cascade: energy transfer towards dissipation scales (small scales)

2. ***Inertial interval***: faraway from forcing and dissipation energy transfer rate is defined by statistical properties of turbulence only

    - Universality for all scales
    - Independency from sources and dissipations

Zakharov 1965
Similar behaviour for wave turbulence

Ideal invariants MHD/HD

MHD Invariants in 2D & 3D

$$E = \frac{1}{2} \int (v^2 + b^2) \; d^n x$$

$$H^c = \frac{1}{2} \int \mathbf{A} \cdot \mathbf{v} \; d^n x$$

Direct cascade:
large scales
towards small
scales
3D & 2D

$$H^M = \frac{1}{2} \int \mathbf{A} \cdot \mathbf{b} \; d^3 x$$

3D

Inverse cascade

$$H^M = \frac{1}{2} \int |\mathbf{A}|^2 \; d^2 x$$

2D

HD invariants in 3D

$$E = \frac{1}{2} \int |\mathbf{u}|^2 \; d^n x$$

$$H^k = \frac{1}{2} \int \mathbf{u} \cdot \boldsymbol{\omega} \; d^n x$$

Direct cascade:
The 3D only

HD Invariants in 2D

$$u = (\partial_y \psi, -\partial_x \psi) \qquad \omega = \nabla \times u = \nabla^2 \psi \hat{\mathbf{z}}$$

$$E = \frac{1}{2} \int |\nabla \psi|^2 \; dx$$

Inverse cascade

$$H^k = \frac{1}{2} \int \omega^2 \; dx$$

Direct cascade

MHD turbulence

MHD turbulence : exhibits strongly anisotropic states $k_\perp \gg k_{||}$

Turbulent cascade in the direction perpendicular to background magnetic field $\mathbf{B}_0$

•(1963) Iroshnikov & (1965) Kraichnan: $E_k \propto k^{-3/2}$
Isotropic wave turbulence in 3D (phenomenology)

Wave turbulence

Statistical description of a large ensemble of weakly interacting dispersive waves

• (2000) Galtier et al. : Anisotropic turbulence in 3D (wave-kinetic approach) $E_k \propto k_\perp^{-2}$

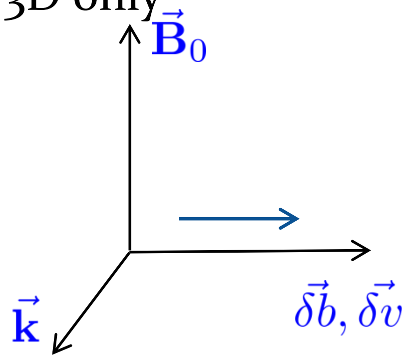
What are the main differences between the turbulent dynamics in 2D et 3D MHD ?

1)Numerical simulations: strong statements about similar behaviour.

2)Analytical results: verification by wave-turbulence approach

Alfvén waves : 2D/3D

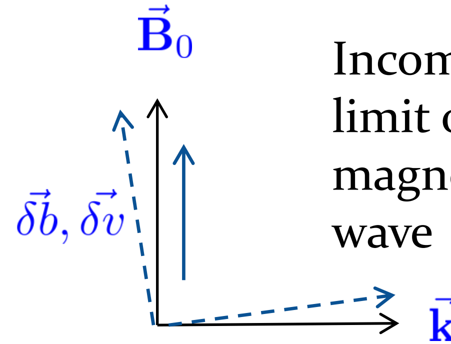
Shear Alfvén waves (SAW): 3D only



Two types of linear modes for incompressible MHD

$$\boxed{\begin{gathered}\epsilon = \frac{b_\perp k_\perp}{B_0 k_{||}} \ll 1 \\[2em] \sigma = \frac{k_\perp}{k_{||}} \gg 1\end{gathered}}$$

Pseudo Alfvén waves (PAW): 3D and 2D



Incompressible limit of slow magneto-sonic wave

Fluctuations of velocity and magnetic field are transverse to $\vec{\mathbf{B}}_0$

Fluctuations of velocity and magnetic field are parallel to $\vec{\mathbf{B}}_0$

3D: leading order of small nonlinearity & strong anisotropy:

- SAW: three wave non-linear interaction, independent on PAW;
- PAW: scattering on SAW, no interaction between themselves;

2D: only PAW exist, how do they interact?

Incompressible 2D MHD

$$\left(\partial_t - sB_0\partial_x + \mathbf{z}^{-s}\cdot\nabla\right)\mathbf{z}^s = -\nabla P_*$$

$$\nabla\cdot\mathbf{z}^s = 0$$

Elsasser variables

$$\mathbf{z}^s = \mathbf{u} + s\mathbf{b}$$

$$s = \pm$$

$$\mathbf{B} = B_0\,\hat{\mathrm{x}} + \mathbf{b}$$

I. Linear dynamics

$$z_j^s \sim e^{i(k_x x + k_y y) - i\omega^s t}$$

Two eigenmodes

$$\omega^+ = -B_0 k_x \qquad \omega^- = B_0 k_x$$

$$\mathbf{u} = -\mathbf{b} \qquad \mathbf{u} = \mathbf{b}$$

Anti parallel — Parallel

II. Fourier transform

$$\mathbf{z}_j^s(\mathbf{x},t) = \sum_{\mathbf{k}} a_j^s(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{x}}$$

$$\hat{P}_*(\mathbf{k}) = -k^{-2}\sum_{\mathbf{k}_1,\mathbf{k}_2}\left(\mathbf{k}_2\cdot\mathbf{a}^s(\mathbf{k}_1,t)\right)\left(\mathbf{k}\cdot\mathbf{a}^s(\mathbf{k}_2;t)\right)$$

$$\times\delta\left(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}\right)$$

$$\mathbf{k} = (k_x,k_y) = (2\pi m_x/L, 2\pi m_y/L)$$

Incompressibility condition $a_x = -\dfrac{k_y}{k_x}a_y$

& Anisotropic limit $k_y \gg k_x$

Wave-kinetic approach. I

1. **Representation of interaction:**
   scales of motion separation:
   - Slow amplitude
   - Fast phase

$$\dot{c}_k^{\pm} = -i\epsilon \sum_{1,2} V_{k12} e^{\pm 2ik_{1x}t} c_1^{\mp} c_2^{\pm} \delta_{12}^k$$

$$V_{k12} \equiv V(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = \frac{(\mathbf{k} \cdot \mathbf{k}_2)\,[\mathbf{k}_1, \mathbf{k}_2]_z}{k\ k_1\ k_2}$$

$$a_x^s(\mathbf{k}, t) = -i\ \epsilon\ \frac{k_y}{k} c_k^s\ e^{-i\omega^s t}$$

Small parameter designes linear and non-linear frequences separation

$$\epsilon = \frac{\omega_{nl}}{\omega_L} = \frac{b\ k_\perp}{B_0\ k_{||}}$$

The approach is applied only if the small parameter can be defined

2. **Itermediate time scale:** $t_L \ll T \ll t_{nl}$
   - The turbulence is enough developed
   - Far away from strong non-linearities

$$t_L = 2\pi/\omega_L$$

$$T = 2\pi/(\epsilon\ \omega_L)$$

$$t_{nl} = 2\pi/(\epsilon^2 \omega_L)$$

Perturbative series solutions

$$c_k^{\pm}(T) = c_k^{(\pm,0)} + \epsilon c_k^{(\pm,1)} + \epsilon^2 c_k^{(\pm,2)} + \dots$$

Wave-kinetic approach. II

3. Energy spectrum: $$E_k^{\pm}(T) - E_k^{\pm}(0) = \left\langle \left| c_k^{(\pm,1)} \right|^2 \right\rangle + \left\langle c_k^{(\pm,0)*} c_k^{(\pm,2)} \right\rangle + \left\langle c_k^{(\pm,0)} c_k^{(\pm,2)*} \right\rangle$$

The initial amplitudes-Gaussian independent variables
Use Wick's rule:

$$\left\langle c_{k1}^{(\pm,0)}\ c_{k2}^{(\pm,0)}\ c_{k3}^{(\mp,0)}\ c_{k4}^{(\mp,0)} \right\rangle = \delta(k_1+k_2)\delta(k_3+k_4) \left\langle \left| c_{k1}^{(\pm,0)} \right|^2 \right\rangle \left\langle \left| c_{k3}^{(\pm,0)} \right|^2 \right\rangle$$

$$\left(c(k)^{(\pm,0)} \right)^* = c(-k)^{(\pm,0)}$$

•Infinite box limit $L \to \infty$

•Small non-linearity $T \to \infty$

At the intermediate time scale small
comparing to physical one, we use: $\left(E_k(T) - E_k(0)\right)/T \sim \dot{E}_k$

Wave-kinetic approach. III

4. Wave kinetic equation for 3 waves interaction:

$$\boxed{\dot{E}_k^{\pm}(k_x,k_y;t) = \pi\epsilon^2 \int V_{k12}^2 E_1^{\mp}(k_{1x}k_{1y};t)\left[E_2^{\pm}(k_{2x},k_{2y};t) - E_k^{\pm}(k_x,k_y;t)\right]\delta\left(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2\right)\delta(2\ k_{1x})d\mathbf{k}_1\ d\mathbf{k}_2}$$

5. Anisotropic limit:

$$V_{k12} = \frac{(\mathbf{k}\cdot\mathbf{k}_2)\left[\mathbf{k}_1,\mathbf{k}_2\right]_z}{k\ k_1\ k_2} \xrightarrow{k_y \gg k_x} V_{12}^k = k_x$$

$$\frac{dE_k^-(k_x,k_y;t)}{dt} = \pi\epsilon^2 k_x^2 \int dk_{1y}\ dk_{2y}\ \delta(k_{1y}+k_{2y}-k_y)\ E_1^+(0,k_{1y};t)\Big(E_2^-(k_{2x},k_{2y},t) - E_k^-(k_x,k_y,t)\Big)$$

- The energy is conserved for each k_x $\qquad \partial_t \int E^{\pm}(k_x,k_y,t)dk_y = 0$
- Decoupling of parallel and perpendicular dynamics
- New « mixte » evolution parameter $\qquad \boxed{\tau = \pi\epsilon^2 k_x^2 t}$

Spectrum study

Separation of parallel and perpendicular spectrum

$$E(k_x, k_y; t) = \mu(k_x)\eta(k_y; t)$$

I. Parallel spectrum: stationary behaviour $\mu(0) = 1$

II. Perpendicular spectrum dynamics:

$$\boxed{\frac{\partial \eta^-(k_y,\tau)}{\partial \tau} = \int_{-\infty}^{\infty} dk_{1y} dk_{2y}\, \delta\left(k_{1y} + k_{2y} - k_y\right) \eta^+\left(k_{1y}, 0\right) \left[\eta^-\left(k_{2y}, \tau\right) - \eta^-\left(k_y, \tau\right)\right]}$$

Great simplification:
Pseudo physical space

$$\mathcal{E}^\pm(y,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \eta^\pm(k_y,\tau) e^{ik_y y} dk_y$$

PDE for spectrum description $$\frac{\partial \mathcal{E}^-(y,\tau)}{\partial \tau} = \mathcal{E}^-(y,\tau)\left(\mathcal{E}^+(y,0) - \mathcal{E}^+(0,0)\right)$$

Mixte evolution parameter

$$\tau = \pi \epsilon^2 k_x^2 t$$

What about locality of turbulence?

I. Kolmogorov solutions

Power law steady state solutions

$$E_k^{\pm} \propto |k_y|^{\alpha^{\pm}}$$

Zakharov transformation

$$k'_{1y} = \frac{k_y k_{1y}}{k_{2y}}, \; k'_{2y} = \frac{k_y^2}{k_{2y}}$$

	2 D	3 D
• Kolmogorov like spectrum	$\alpha^+ + \alpha^- = -2$	$\alpha^+ + \alpha^- = -4$
• Balanced turbulence case	$\alpha^+ = \alpha^- = -1$	$\alpha^+ = \alpha^- = -2$
•Convergence zone	1. No convergence zone; 2. Existence of the limit point with logarithmic divergence for the balanced turbulence case.	$-3 < \alpha_{\pm} < -1$

• **Locality ?**

•**Logarithmic corrections of Kraichnan?**

Forcing and dissipation

- Balanced turbulence case $\mathcal{E}^{+} = \mathcal{E}^{-}$

- Forcing and dissipations $\dfrac{\partial \mathcal{E}(y,\tau)}{\partial \tau} = \mathcal{E}(y,\tau)\left(\mathcal{E}(y,0) - \mathcal{E}(0,0) + \sigma_d\right) + \mathcal{F}(y)$

Main interest :	a) small scales response	$k \gg k_f$	$k \gg k_0$
	b) stationary solutions	$\tau \to \infty$	

Initial conditions

An example:

$$\eta(k_y,0) = A\delta(k_y - k_0) + A\delta(k_y + k_0)$$

$$\Downarrow$$

$$\mathcal{E}(y,0) = 2A\cos(k_0 y) \qquad \mathcal{E}(0,0) = 2A$$

I. Uniform friction

$$\partial_\tau \mathcal{E}(y,\tau) = \mathcal{E}(y,\tau)\ \left(2A\left[\cos(k_0 y)-1\right]-\sigma_d\right)+\sigma_f e^{-\frac{k_f^2 y^2}{2}}$$

Inertial interval

$$k \gg k_0 \ \Rightarrow\ \cos(k_0 y) = 1 - 1/2(k_0 y)^2 + \ldots$$

$$k \gg k_f \ \Rightarrow\ e^{-\frac{k_f^2 y^2}{2}} \approx 1$$

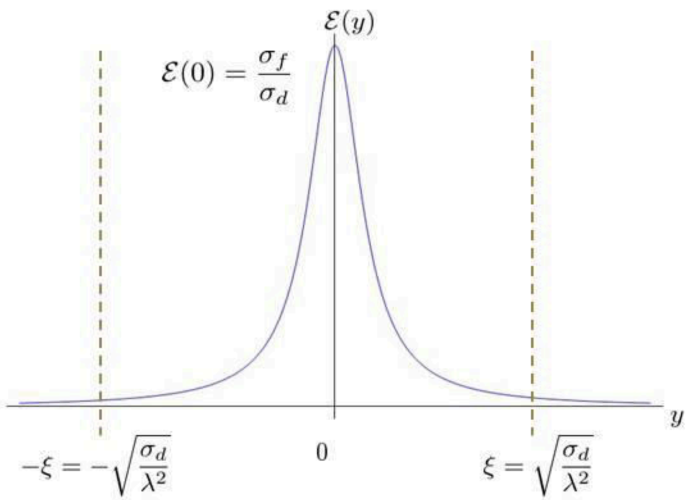
The steady state solution

$$\mathcal{E}(y,\infty) = \frac{\sigma_f}{\sigma_d + \lambda y^2}$$

$$\xrightarrow{\lambda = A k_0^2}$$

$$\xi = \sqrt{\frac{\sigma_d}{\lambda^2}} \to 0$$

$$\mathcal{E}(y,\infty) = \pi \frac{\sigma_f}{\sqrt{\lambda\ \sigma_d}}\delta(y)$$



No energy cascade; But global transfer; flat spectrum in physical space

At scales $y \gg \xi \Leftrightarrow k \ll \frac{1}{\xi}$

$$\boxed{\eta(k_y,\infty) = \frac{1}{2}\frac{\sigma_f}{\sqrt{\lambda\ \sigma_d}}}$$

The steady state solution depends on forcing, dissipation and initial conditions

II. Viscous friction: stationary solutions

$$\frac{\partial \mathcal{E}(y,\tau)}{\partial \tau} = 2A\left(\cos(k_0 y) - 1\right)\mathcal{E}(y,\tau) + \nu \frac{\partial^2 \mathcal{E}(y,\tau)}{\partial y^2} + \sigma_f e^{-\frac{k_f y^2}{2}}$$

The change of variables

$$\tilde{y} = y\left(\frac{\lambda}{\nu}\right)^{\frac{1}{4}}, \ \tilde{\mathcal{E}} = \frac{\sqrt{\lambda\nu}}{\sigma_f}\mathcal{E}$$

The asymptotes

$\tilde{y} \gg 1$ $\qquad \tilde{\mathcal{E}}(\tilde{y},\infty) = 1/\tilde{y}^2$

$\tilde{y} \ll 1$ $\qquad \tilde{\mathcal{E}}(\tilde{y},\infty) = C - \tilde{y}^2$

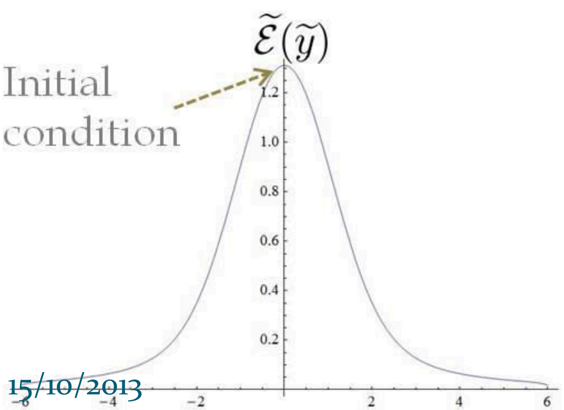
The « Master equation »:
The equation of the *Hyperbolic Cylinder*

$$\frac{d^2\tilde{\mathcal{E}}(\tilde{y},\infty)}{d\tilde{y}^2} - \tilde{y}^2\tilde{\mathcal{E}}(\tilde{y},\infty) + 1 = 0$$

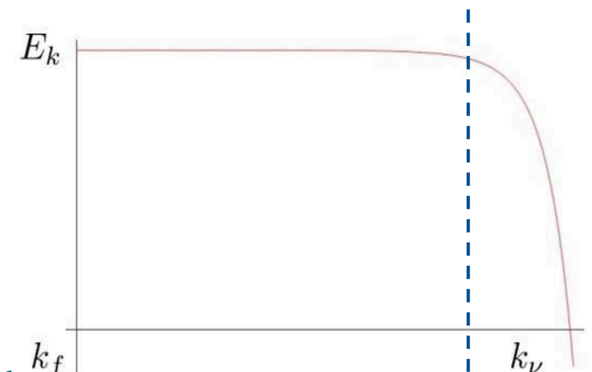
$$\eta(k_y,\infty) = C\sigma_f\left(\frac{\nu}{\lambda^3}\right)^{1/4}$$

$$k_0, k_f \ll k \ll k_\nu = (\lambda/\nu)^{1/4}$$

Initial condition



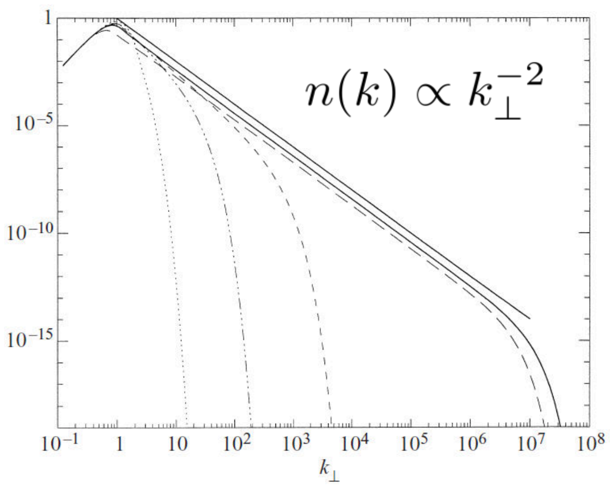
Qualitative spectrum



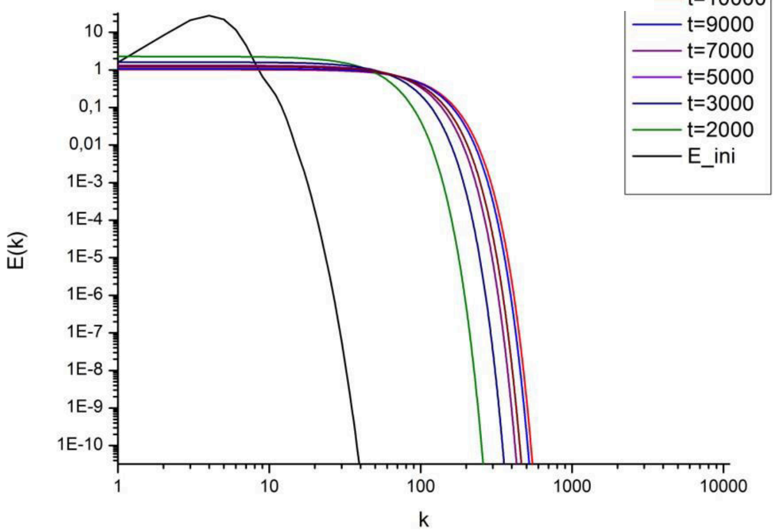
Numerical results 2D/3D

Viscous friction

3D



2D



$$\mathcal{F}_y(k_y) = \mathcal{F}_{force} - \sigma_\nu k_y^2$$

$$\mathcal{F}_{force} = 3 \times 10^{-4}/k$$

$$\sigma_\nu = 10^{-6} \qquad k \in [3, 9]$$

3D case: interaction of SAW
Galtier et al. JPP 2000

Absence of cascade; suppression of energy transfer on the small scales

Beyond wave turbulence

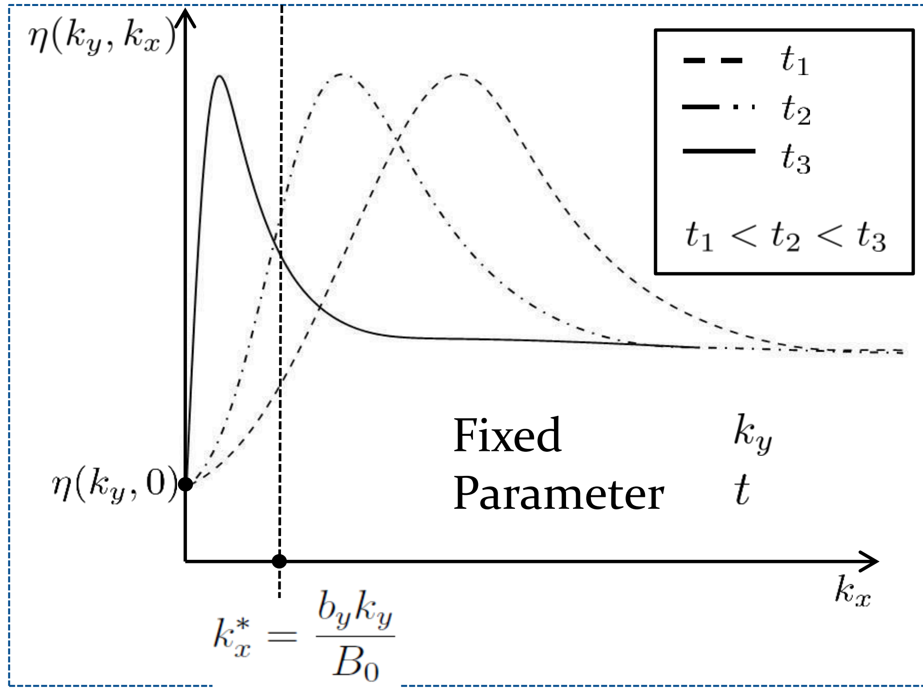
Limit of the wave-kinetic approach applicability $\epsilon = \frac{t_{lin}}{t_{NL}} = \frac{b_\perp k_\perp}{B_0 k_{||}} \ll 1$

What happens for values of parallel wave number close to zero?

3D: Requirement of a sufficient spectrum smoothness for $k_{||} = 0$

2D: Smoothness of the spectrum near $k_{||} = 0$ is asymptotically broken in time.

Main difference with 3D case: dependence on $k_{||}$ in the perpendicular spectrum via self-similar evolution parameter $\tau = k_{||}^2 t$



- Spectrum narrowing at large time scales
- Important spectrum evolution in forbidden region
- Breakdown of the wave kinetic equation for late evolution $t \sim (k_x^*)^{-2}$

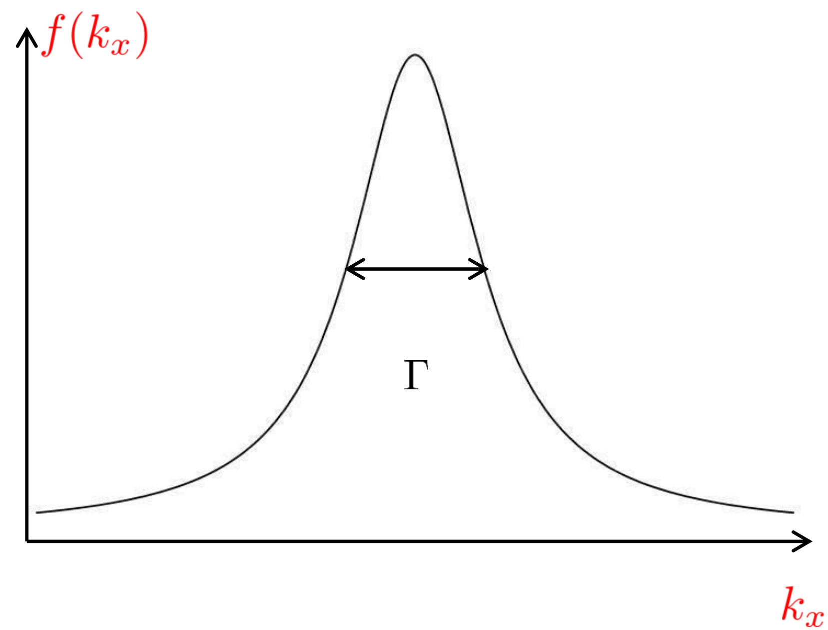
Can we construct a spectrum description beyond the wave-turbulence approach?

Qualitative approach

Arguments for qualitative approach construction:

- Three waves interactions are **never exactly resonant**
- **Resonance broadening:**
  - The quasiresonant frequencies within the viscinity of the resonance are involved in wave-interactions
  - Non-important in the case of smooth spectrum behaviour but accounts in the situation with strong gradients

- **In the strong gradient zone:** $\Delta k \sim k_x^*$
  - Replace the delta-function by smoothing function f(k_x)
  - Acts a a filter for strong gradients



Spectrum behaviour beyond WT

Small scales: $k \gg k^*$

•Spectrum remains slowly varying
•Kinetic equation can be amended with replacing the $\delta(2k_{1x}) = \delta(\omega_{1k} + \omega_{2k} - \omega_k)$ by $f(k_{1x})$

$$n^{\pm}(0, k_y, 0) = \int n^{\pm}(k_{1x}, k_{1y}, \tau)\, \delta(k_{1x}) dk_{1x}$$

$$\Downarrow$$

$$\langle n^{\pm} \rangle (k_y, \tau) = \int n^{\pm}(k_{1x}, k_{1y}, \tau)\, f(k_{1x}) dk_{1x}$$

Large scales: $k \sim k^*$

•Breakdown of spectrum value conservation at $k_{||} = 0$

•Use $\eta(k_y, 0)$ change for the

$$\Downarrow$$

Gradient smoothing process

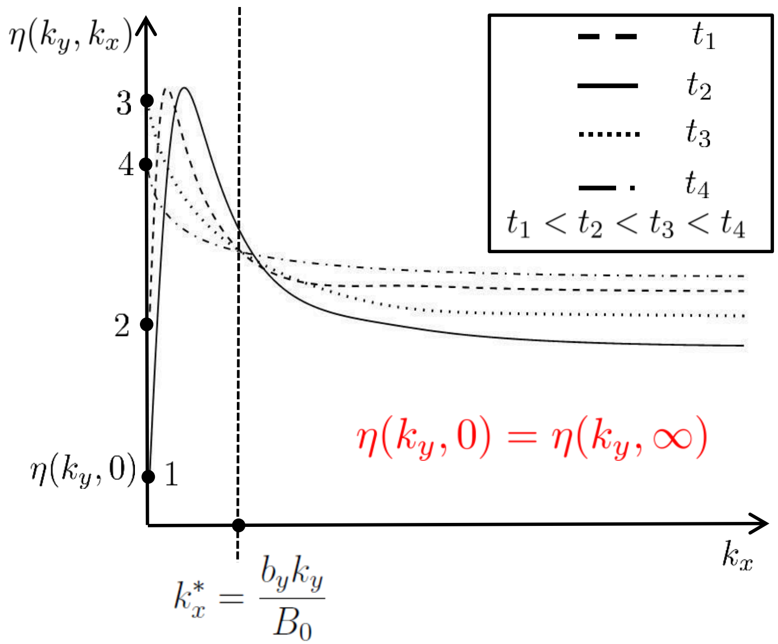
Gradient smoothing process

Gradient smoothing process:

Four iterations towards spectrum stabilization:

•**Point 1:** The gradient of the spectrum in the viscinity of $k_{||} = 0$ is positive;
the initial value increases and reaches
• **Point 2:** it crosses the max and moves at
•**Point 3** with negative slope of spectrum;
The initial value decreases and arrives at the **Point 4;**

•This process takes place until spectrum is stabilized



Wiping out of the k_x dependence by self-similar shrinking of spectrum

Summary of spectrum evolution

(Fixed k_y)

- **The early stage**: $t \ll (k_x^*)^{-2}$ evolution is described by three wave kinetic equation

- **The advanced stage**: $t \sim (k_x^*)^{-2}$ the kinetic equation is broken down by its own evolution

- **Large time scales**: $t \gg (k_x^*)^{-2}$
  - smoothing of strong gradients in $k_{||} = 0$
  - re-emergence of the kinetic equation with steady state

$$\eta(k_y, 0) = \eta(k_y, \infty)$$

The wave-kinetic equation for the steady state

Re-established wave-kinetic equation on the pseudo-physical space

$$\mathcal{E}(y,0) = \mathcal{E}(y,\infty) = \mathcal{E}(y)$$

$$\mathcal{E}^2(y) - \mathcal{E}(y)\left(\mathcal{E}(0) + \sigma_d\right) + \mathcal{F}(y) = 0$$

Initial value is defined by the equation itself

$$y = 0 \Rightarrow \mathcal{E}(0) = \mathcal{F}(0)/\sigma_d$$

$$\mathcal{E}(0) = \int \eta(k_y) dk_y > 0$$

$$\mathcal{E}(y) = \frac{1}{2}\left(\sigma_d + \mathcal{F}(0)/\sigma_d\right) \pm \frac{1}{2}\left(\left(\sigma_d + \mathcal{F}(0)/\sigma_d\right)^2 - 4\mathcal{F}(y)\right)^{1/2}$$

No inertial range because the steady state width directly depends on forcing width.

Strongly non-local interaction;

$$\mathcal{E}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \eta^{\pm}(k_y) e^{ik_y y} dk_y$$

Steady state solution at the small scales

$$\mathcal{E}(y) = \frac{1}{2}\left(\sigma_d + \mathcal{F}(0)/\sigma_d\right) \pm \frac{1}{2}\left(\left(\sigma_d + \mathcal{F}(0)/\sigma_d\right)^2 - 4\mathcal{F}(y)\right)^{1/2}$$

Decaying forcing (i.e.Gaussian)

$$\lim_{y\to\infty} \mathcal{F}(y) \to 0$$

$-$ $\mathcal{F}(0) < \sigma_d$

$$\lim_{y\to\infty} \mathcal{E}(y) \to 0$$

$+$ $\mathcal{F}(0) > \sigma_d$

$$\lim_{y\to\infty} \mathcal{E}(y) \to \sigma_d + \mathcal{F}(0)/\sigma_d$$

Condensation into the state with delta function at $k_y = 0$

Physically meaningful situation: dissipation dominates forcing at small scales: no delta function on physical space;
Flat spectrum solutions; suppression of energy transfer on small scales

Conclusions

- **Simple structure of the wave-kinetic equation in 2D:** analytical investigation of spectrum dynamics

- **Comparaison with 3D case**
  - **Absence of turbulence locality:**
    - Absence of Kolmogorov-like solutions
    - Global energy transfer in two limiting cases: uniform friction and viscous friction; no energy cascade
    - **Should be verified via DNS**
  - **Beyond wave-turbulence**
    - Qualitative approach
    - Breakdown of wave-kinetic equation due to the self-similar dependence on parallel spectrum
    - Re-emmergence of the wave-kinetic equation at the large time scales
    - No energy cascade
- **More details [Tronko et al, PRE 2013]**