Asymptotic preserving methods for the BGK-Vlasov-Poisson system in the quasi-neutral and fluid limits

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Outline

The BGK-Vlasov-Poisson model

Unknowns f : electron distribution function, ϕ : electric potential

One species model for clarity $x \in \Omega \subset \mathbb{R}^d$, $v \in \mathbb{R}^d$, $t \geq 0$

$$(S_{\varepsilon,\lambda}) \begin{cases} \partial_t f + v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f = \dfrac{1}{\varepsilon} \dfrac{1}{\tau(\rho, T)} (M[f] - f) = \dfrac{1}{\varepsilon} Q(f), \\ \lambda^2 \Delta \phi = \rho - \rho_i, \end{cases}$$

$$M[f] = \frac{\rho}{(2\pi T)^{d/2}} \exp\left(\frac{-|v-u|^2}{2T}\right), \quad \begin{pmatrix} \rho \\ \rho u \\ \dfrac{d}{2}\rho T + \dfrac{\rho |u|^2}{2} \end{pmatrix} = \int \begin{pmatrix} 1 \\ v \\ \dfrac{|v|^2}{2} \end{pmatrix} f \, dv,$$

Data ρ_i : Constant ion density $\qquad \tau(\rho, T)$: scaled relaxation time

λ : scaled Debye length $= \dfrac{\text{Debye length}}{\text{size of the domain}}$

ε : Knudsen number $= \dfrac{\text{mean free path}}{\text{size of the domain}}$

Overall idea

Multi-scale model $S_{\varepsilon,\lambda}$ depending on 2 parameters

ε, λ can : be very small in some regions ➠ microscopic scales
be of order 1 in other ones ➠ macroscopic scales
take all the values between 1 and small values elsewhere

Difficulties :

- Explicit schemes : stable and consistent iff micro. scales are resolved
- Implicit schemes : unconditionally stable and consistent but non linear

➠ Huge cost

A solution : Use a scheme preserving the limits $\lambda, \varepsilon \to 0$

- Mesh independent of λ, ε : **Asymptotic stability**.
- Recover an approximate solution of $S_{\varepsilon,0}$, $S_{0,\lambda}$ or $S_{0,0}$ if ε and/or $\lambda \ll 1$: **Asymptotic consistency.**

Both properties $\Rightarrow$ Asymptotic preserving scheme (AP)

([S.Jin] kinetic $\to$ Hydro)

Outline

The fluid limit of the BGK-Vlasov-Poisson system

Formally passing to the limit $\varepsilon \to 0$ gives

$$(S_{0,\lambda}) \begin{cases} M[f] = f, \\ \partial_t f + v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \frac{1}{\tau(\rho, T)} (M[f] - f), \\ \lambda^2 \Delta \phi = \rho - \rho_i, \end{cases}$$

Taking the moments of Vlasov$\Rightarrow$ Euler equations

$$(S_{0,\lambda}) \begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\ \partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x p = \rho \nabla_x \phi, \\ \partial_t E + \nabla_x \cdot ((E + p) u) = \rho u \cdot \nabla_x \phi, \\ \lambda^2 \Delta \phi = \rho - \rho_i, \end{cases}$$

$$\text{with } p = \frac{2}{d} \left(E - \frac{\rho |u|^2}{2} \right).$$

The QN limit of the BGK-Vlasov-Poisson system

Formally passing to the limit $\lambda \to 0$ gives

$$(S_{\varepsilon,0}) \begin{cases} \partial_t f + v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f = \dfrac{1}{\varepsilon} Q(f), \\ 0 = \rho - \rho_i, \end{cases}$$

➠ ϕ is determined by the constraint $\rho = \rho_i$

To recover an explicit eq. for ϕ we must use the moment eqs of Vlasov

Moment eqs of Vlasov with $\rho u = \int f v \, dv, \qquad S = \int f v \otimes v \, dv.$

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\ \partial_t (\rho u) + \nabla_x S = \rho \nabla_x \phi, \end{cases} \Leftrightarrow \begin{cases} \nabla_x \cdot (\rho u) = 0, \\ \partial_t (\rho u) + \nabla_x S = \rho \nabla_x \phi, \end{cases}$$

$$\Rightarrow \nabla_x \cdot (\rho \nabla_x \phi) = \nabla_x^2 : S \qquad \text{explicit eq. for } \phi$$

The QN limit of the BGK-Vlasov-Poisson system

Reciprocally

$$\nabla_x \cdot (\rho \nabla_x \phi) = \nabla_x^2 : S \quad \Rightarrow \quad \rho(x,t) = \rho(x,0) - t \nabla_x \cdot (\rho u)(x,0)$$

The reformulated BGK-Vlasov-QN system

$$(RS_{\varepsilon,0}) \begin{cases} \partial_t f + v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f = \dfrac{1}{\varepsilon} Q(f), \\ \nabla_x \cdot (\rho \nabla_x \phi) = \nabla_x^2 : S, \end{cases}$$

$$(RS_{\varepsilon,0}) \Leftrightarrow (S_{\varepsilon,0}) \quad \text{iff} \quad \rho(x,0) = \rho_i \quad \text{and} \quad \nabla_x \cdot (\rho u)(x,0) = 0.$$

➟ The reformulated system does not project the density on the quasi-neutral state.

The joint QN and fluid limits

The joint limit $(\lambda, \varepsilon) \to (0,0)$ **gives**

$$(S_{0,0}) \begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\ \partial_t(\rho u) + \nabla_x \cdot S = \rho \nabla_x \phi, \\ \partial_t E + \nabla_x \cdot ((E+p)u) = \rho u \cdot \nabla_x \phi, \\ 0 = \rho - \rho_i, \end{cases} \quad \text{with} \quad \begin{cases} S = (\rho u \otimes u) + p\, Id \\ p = \dfrac{2}{d}\left(E - \dfrac{\rho |u|^2}{2}\right) \end{cases}$$

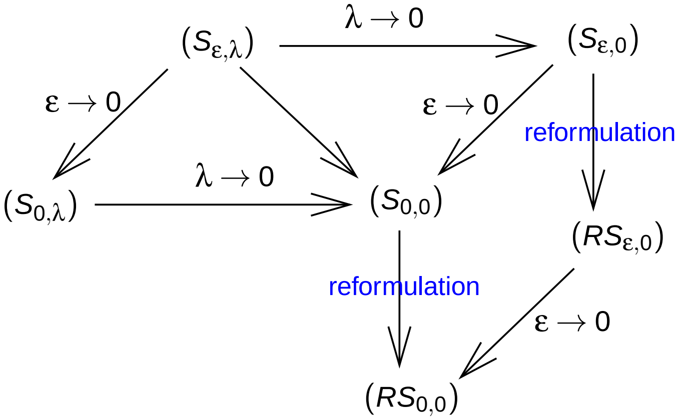
This system can be also reformulated

$$(RS_{0,0}) \begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\ \partial_t(\rho u) + \nabla_x \cdot S = \rho \nabla_x \phi, \\ \partial_t E + \nabla_x \cdot ((E+p)u) = \rho u \cdot \nabla_x \phi, \\ \nabla_x \cdot (\rho \nabla_x \phi) = \nabla_x^2 : S, \end{cases} \quad \Leftrightarrow \quad (S_{0,0})$$

iff $\begin{cases} \rho(x,0) = \rho_i \\ \nabla_x \cdot (\rho u)(x,0) = 0 \end{cases}$

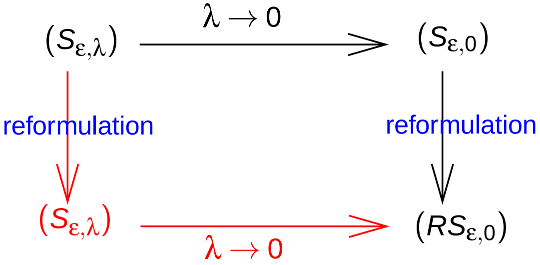
All the limits

Summary



Reformulation of the quasi-neutral limits

Summary



It is possible to complete the diagram following the same ideas.

Start from the BGK-Vlasov-Poisson system

$$(S_{\varepsilon,\lambda}) \begin{cases} \partial_t f + v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f = \dfrac{1}{\varepsilon} Q(f), \\ \lambda^2 \Delta \phi = \rho - \rho_i, \end{cases}$$

Reformulation of the BGK-Vlasov-Poisson system

Work on the moment eqs

$$(S_{\varepsilon,\lambda}) \Rightarrow \begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\ \partial_t(\rho u) + \nabla_x S = \rho \nabla_x \phi, \\ \lambda^2 \Delta \phi = \rho - \rho_i, \end{cases} \Rightarrow \begin{cases} \partial_{tt}^2 \rho + \partial_t \nabla_x \cdot (\rho u) = 0, \\ \nabla_x \cdot \partial_t(\rho u) + \nabla_{xx}^2 : S = \nabla_x \cdot \Big(\rho \nabla_x \phi\Big), \\ \lambda^2 \partial_{tt}^2 \Delta \phi = \partial_{tt}^2 \rho, \end{cases}$$

$$\Rightarrow \quad \lambda^2 \partial_{tt}^2 \Delta \phi + \nabla_x \cdot \Big(\rho \nabla_x \phi\Big) = \nabla_{xx}^2 : S.$$

The reformulated BGK-Vlasov-Poisson system

$$(RS_{\varepsilon,\lambda}) \begin{cases} \partial_t f + v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f = \dfrac{1}{\varepsilon} Q(f), \\ \lambda^2 \partial_{tt}^2 \Delta \phi + \nabla_x \cdot \Big(\rho \nabla_x \phi\Big) = \nabla_{xx}^2 : S. \end{cases} \Leftrightarrow (S_{\varepsilon,\lambda})$$

$$\text{iff} \begin{cases} \lambda^2 \Delta \phi(x,0) = \rho(x,0) - \rho_i, \\ \lambda^2 \partial_t \Delta \phi(x,0) = -\nabla_x \cdot (\rho u)(x,0). \end{cases}$$

Numerical point of view

$$\partial_t f + v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f = \frac{1}{\varepsilon} Q(f)$$

- Implicit treatment of $Q(f)$ is necessary otherwise $\Delta t \leq \varepsilon$

$$\lambda^2 \partial_{tt}^2 \Delta\phi + \nabla_x \cdot \left(\rho \nabla_x \phi\right) = \nabla_{xx}^2 : S.$$

- Harmonic oscillator eq. on $\Delta\phi$: implicit treatment necessary
  - Explicit treatment of $\phi \quad \Rightarrow \quad$ conditional stability : $\Delta t \leq \lambda$
- Consistency properties
  - Does not degenerate when $\lambda \to 0$ and reduces to $(RS_{\varepsilon,0})$ if $\lambda = 0$
- $(RS_{\varepsilon,\lambda}) \Leftrightarrow (S_{\varepsilon,\lambda})$ iff $\begin{cases} \lambda^2 \Delta\phi(x,0) = \rho(x,0) - \rho_i, \\ \lambda^2 \partial_t \Delta\phi(x,0) = -\nabla_x \cdot (\rho u)(x,0). \end{cases}$

  With not well prepared initial conditions, the scheme must project the density on the state ρ_i

Outline

Asymptotic preserving scheme in the fluid limit

AP schemes in the hydrodynamic limit of the Vlasov eq.

- No electric field but more general collision operator (Boltzmann)

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} Q(f)$$

- References
  - S. Pieraccini & G. Puppo (J. Sci Comput. 2007),
  - M. Bennoune & M. Lemou & L. Mieussens (JCP 2008),
  - F. Filbet & S. Jin (JCP 2010),
  - B. Yan & S. Jin (SIAM J. Sci Comput. 2012)
  - G. Dimarco & L. Pareschi (SIAM J. Num Anal. 2013)

Existing AP scheme in the fluid limit

Idea in the case of the BGK Operator if f^n is known

$$\frac{f^{n+1}-f^n}{\Delta t}+v\cdot\nabla_x f^n=\frac{1}{\varepsilon\tau(\rho^{n+1},T^{n+1})}\left(M[f^{n+1}]-f^{n+1}\right),$$

- M depends only on the moments of f $\Rightarrow$ can be solved explicitly
- Taking the moments of Vlasov with respect to the velocity

$$\frac{\rho^{n+1}-\rho^n}{\Delta t}+\nabla_x\cdot\int v\, f^n\,dv=0,\qquad \frac{E^{n+1}-E^n}{\Delta t}+\nabla_x\cdot\int\frac{|v|^2}{2}v\, f^n\,dv=0,$$

$$\frac{(\rho u)^{n+1}-(\rho u)^n}{\Delta t}+\nabla_x\cdot\int v\otimes v\, f^n\,dv=0,\quad T^{n+1}=\frac{d}{2}\left(\frac{E^{n+1}}{\rho^{n+1}}-\frac{|u|^2}{2}\right),$$

- Δt independent of ε $\Rightarrow$ asymptotically stable
- if $\varepsilon=0$, $f^{n+1}=M[f^{n+1}]$ $\Rightarrow$ asymptotically consistent

Classical scheme in the quasi-neutral limit

On isentropic Euler system for clarity

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\ \partial_t (\rho u) + \nabla_x \cdot S = \rho \nabla_x \phi, \qquad \text{with} \quad S = \rho u \otimes u + \rho^\gamma Id, \quad \gamma > 1 \\ \lambda^2 \Delta \phi = \rho - \rho_i, \end{cases}$$

Classical scheme

$$\begin{cases} \dfrac{\rho^{n+1} - \rho^n}{\Delta t} + \nabla_x \cdot (\rho u)^n = 0, \\ \dfrac{(\rho u)^{n+1} - (\rho u)^n}{\Delta t} + \nabla_x \cdot S^n = \rho^n \nabla_x \phi^{n+1}, \\ \lambda^2 \Delta \phi^{n+1} = \rho^{n+1} - \rho_i, \end{cases}$$

[S. Fabre, JCP 92] stable scheme iff $\Delta t \leq \lambda$

Existing AP scheme in the quasi-neutral limit

AP scheme [Crispel, Degond,V, 07] use the reformulated system

$$\begin{cases} \rho^{n+1} - \rho^n + \Delta t\, \nabla_x \cdot (\rho\, u)^{n+1} = 0, \\ (\rho\, u)^{n+1} - (\rho\, u)^n + \Delta t\, \nabla_x \cdot S^n = \Delta t\, \rho^n\, \nabla_x \phi^{n+1}, \\ \dfrac{\lambda^2 \Delta\phi^{n+1} - 2\,\lambda^2 \Delta\phi^n + \lambda^2 \Delta\phi^{n-1}}{\Delta t^2} + \nabla_x \cdot \left(\rho^n \nabla_x \phi^{n+1}\right) = \nabla^2_{xx} : S^n, \end{cases}$$

- Δt independent of λ $\Rightarrow$ asymptotically stable [Degond, Liu, V, 08]
- Initially, 2 resolutions of Poisson are necessary,

  ➠ we recover the conditions $\begin{cases} \lambda^2\, \Delta\phi^0 = \rho^0 - \rho_i\,, \\ \lambda^2\, (\partial_t \Delta\phi)^0 = -\nabla_x \cdot (\rho\, u)^0. \end{cases}$
- Can be reduced at 1 initial resolution of Poisson, changing by

$$\frac{\lambda^2 \Delta\phi^{n+1} - 2\,(\rho^n - \rho_i) + \rho^{n-1} - \rho_i}{\Delta t^2} + \nabla_x \cdot \left(\rho^n \nabla_x \phi^{n+1}\right) = \nabla^2_{xx} : S^n$$

Drawbacks of the AP scheme in the QN limit

Very sensitive to the choice of initial conditions

- Due to the 1 or 2 initial iterations of the Poisson eq.
- Can be improved remarking that the scheme is equivalent to

$$\begin{cases} \rho^{n+1} - \rho^n + \Delta t\, \nabla_x \cdot (\rho u)^{n+1} = 0, \\ (\rho u)^{n+1} - (\rho u)^n + \Delta t\, \nabla_x \cdot S^n = \Delta t\, \rho^n \nabla_x \phi^{n+1}, \\ \lambda^2 \Delta \phi^{n+1} = \rho^{n+1} - \rho_i, \end{cases}$$

➠ Using the mass and momentum eqs

$$\lambda^2 \Delta \phi^{n+1} = \rho^n - \Delta t\, \nabla_x \cdot (\rho u)^{n+1} - \rho_i,$$

$$= \rho^n - \rho_i - \Delta t\, \nabla_x \cdot (\rho u)^n + \Delta t^2 \nabla^2_{xx} : S^n - \Delta t^2 \nabla_x \cdot \left(\rho^n \nabla_x \phi^{n+1} \right).$$

➠ Gives a new discretization of the reformulated Poisson eq.

$$\lambda^2 \Delta \phi^{n+1} + \Delta t^2 \nabla_x \cdot \left(\rho^n \nabla_x \phi^{n+1} \right) = \rho^n - \rho_i - \Delta t\, \nabla_x \cdot (\rho u)^n + \Delta t^2 \nabla^2_{xx} : S^n$$

Drawbacks of the AP scheme in the QN limit

Extension to the Vlasov-Poisson eqs.

- PIC schemes [Degond, Deluzet, Navoret, 2006],
  [Degond, Deluzet, Navoret, Sun, V, 2010]
- Lagrangian schemes
  [Belaouar, Crouseilles, Degond, Sonnendrücker, 2009]
- Eulerian schemes

$$\frac{f^{n+1} - f^n}{\Delta t} + v \cdot \nabla_x f^{n+1} + \nabla_x \phi^{n+1} \cdot \nabla_v f^n = 0,$$

➠ Yields a linear system of size the mesh in v $\Rightarrow$ Huge cost

- Goals of our work :
  - Construct an Eulerian AP scheme in hydrodynamic and QN limits but with explicit treatments of the Vlasov transport terms in x and v

Our new scheme

Isentropic Euler-Poisson system for clarity

$$\begin{cases} \dfrac{\rho^{n+1} - \rho^n}{\Delta t} + \nabla_x \cdot (\rho u)^{n} = 0, \\ \dfrac{(\rho u)^{n+1} - (\rho u)^n}{\Delta t} + \nabla_x \cdot S^n = \rho^n \nabla_x \phi^{n+1}, \\ \lambda^2 \Delta \phi^{n+1} + \Delta t^2 \nabla_x \cdot \left(\rho^n \nabla_x \phi^{n+1}\right) = \rho^{n+1} - \rho_i - \Delta t \nabla_x \cdot (\rho u)^n + \Delta t^2 \nabla_{xx}^2 : S^n \end{cases}$$

In 1-D

- Fourier stability analysis on the linearized system

  $\Rightarrow$ Δt independent of λ

- Num. comparison to the classical scheme in the non linear case

  Non QN perturbation of the uniform solution by the density :

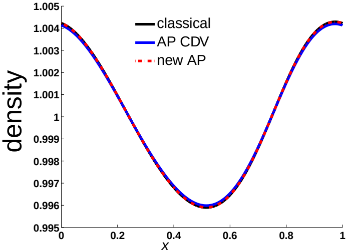
$$\rho = \rho_i = 1, \quad u = 1, \quad \partial_x \phi = 0.$$

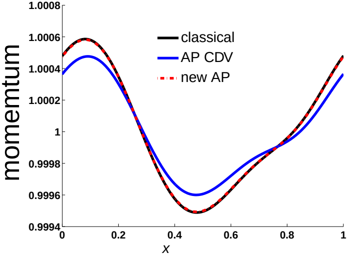
Euler-Poisson, classical & AP schemes $\lambda^2 = 1$

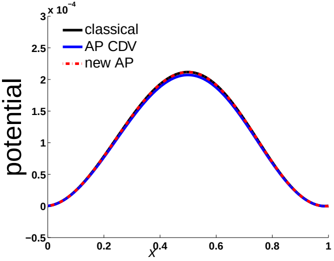
500 cells

$\Delta t \approx 3,5 \cdot 10^{-4} < \lambda$

$T_{max} = 10 \approx 28\,000\,\Delta t$







Euler-Poisson, classical & AP schemes $\lambda^2 = 10^{-5}$

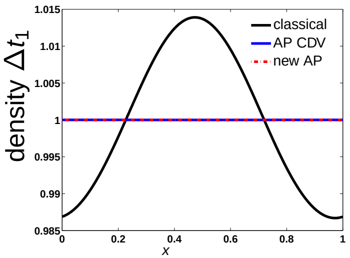
500 cells

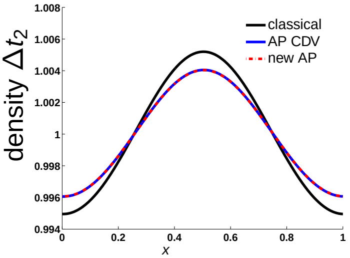
$$\Delta t_1 \approx 2,6 \cdot 10^{-4} < \lambda$$

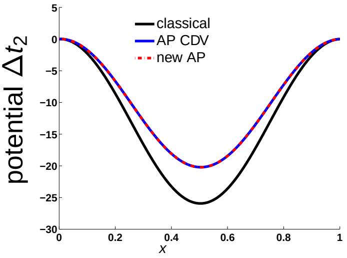
$$T_{max} = 1 \approx 3500\,\Delta t_1$$

$$\Delta t_2 = 10^{-6}$$

$$T_{max} = 1 \approx 200\,000\,\Delta t_2 < \lambda$$





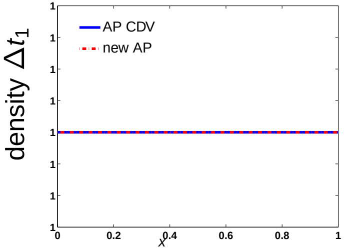


Euler-Poisson, classical &AP schemes $\lambda^2 = 10^{-10}$

500 cells

$$\Delta t_1 \approx 4,4 \cdot 10^{-4} > \lambda$$

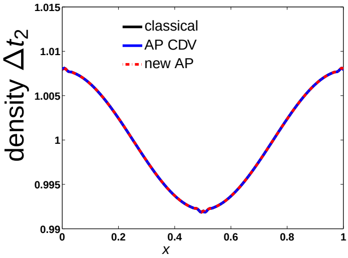
$$T_{max} = 0.1 \approx 230\,\Delta t_1$$

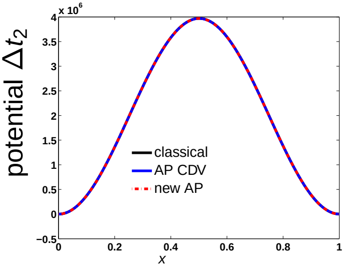


1000 cells

$$\Delta t_2 = 10^{-10}$$

$$T_{max} = 10^{-3} \approx 10^7\,\Delta t_2 < \lambda$$





BGK-Vlasov-Poisson system

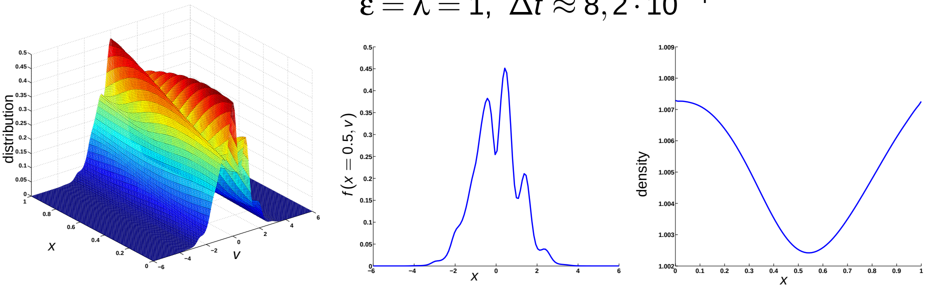
Extension to Vlasov-Poisson, coupling with the BGK AP scheme

$$\begin{cases} \dfrac{f^{n+1}-f^n}{\Delta t} + v\cdot\nabla_x f^n + \nabla_x \phi^{n+1}\cdot\nabla_v f^n = \dfrac{1}{\varepsilon\,\tau^{n+1}}\Big(M[f^{n+1}] - f^{n+1}\Big), \\ \lambda^2 \Delta\phi^{n+1} + \Delta t^2\,\nabla_x\cdot\Big(\rho^n\,\nabla_x\phi^{n+1}\Big) = \rho^{n+1} - \rho_i - \Delta t\,\nabla_x\cdot(\rho\,u)^n + \Delta t^2\,\nabla^2_{xx}:S^n. \end{cases}$$

Initially : Maxwellian distribution with quasi-neutral moments.

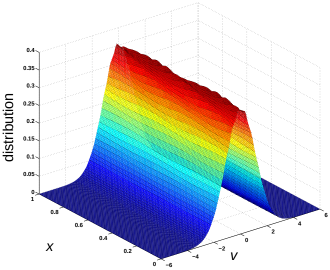
Perturbation of the density and momentum.

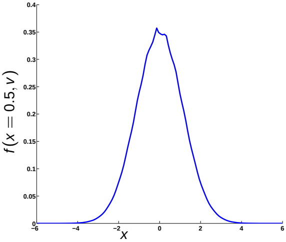
$$x\in[0,1],\quad \Delta x = 1/100,\ v\in[-6,6],\ \Delta v = 1/128$$
$$\varepsilon = \lambda = 1,\ \Delta t \approx 8,2\cdot 10^{-4}$$

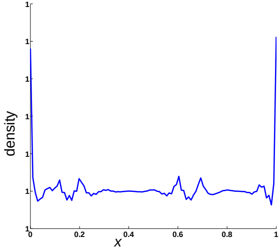


BGK-Vlasov-Poisson system

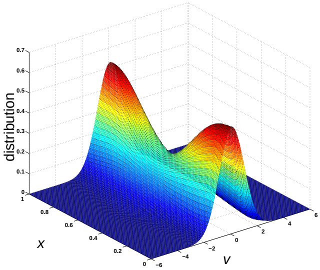
$$\varepsilon = 1, \lambda = 10^{-5},\ \Delta t \approx 8,25 \cdot 10^{-4} > \lambda$$

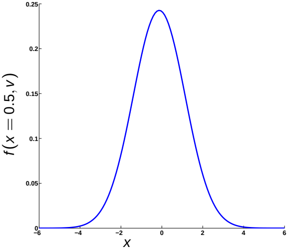


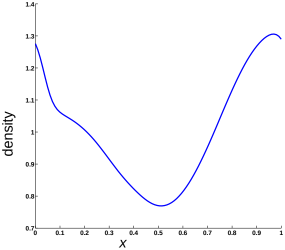




$$\varepsilon = 10^{-5}, \lambda = 1,\ \Delta t \approx 8,25 \cdot 10^{-4} > \varepsilon$$

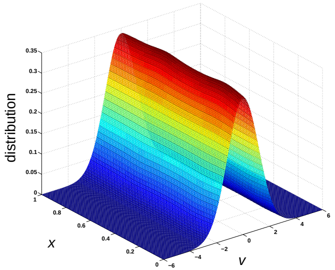


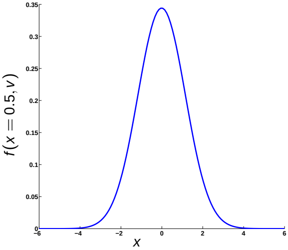


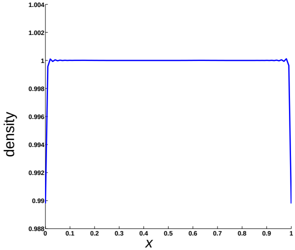


BGK-Vlasov-Poisson system

$$\varepsilon = \lambda = 10^{-5}, \;\; \Delta t \approx 7{,}1 \cdot 10^{-4} > \lambda, \varepsilon$$







Works in progress and perspectives

Works in progress

- Two species model (ions and electrons)

Perspectives

- Multi-dimensional simulations
- High order schemes preserved in the limits
  $\Rightarrow$ asymptotically accurate schemes
- More general collision operator : Boltzmann operator