Kinetic over-relaxation and boundary conditions

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1 Over-relaxation and kinetic scheme

The Jin-Xin method [3] is a simple way to approximate a hyperbolic system of unknown $\mathbf{u}(x,t)$ of the form:

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0. \tag{1}$$

It consists in solving a simpler set of kinetic equations of unknowns $\mathbf{w}(x,t)$ and $\mathbf{z}(x,t)$:

$$\partial_t \mathbf{w} + \partial_x \mathbf{z} = 0, \tag{2}$$

$$\partial_t \mathbf{z} + \lambda^2 \partial_x \mathbf{w} = 0, \tag{3}$$

where λ is a constant speed satisfying a subcharacteristic condition. In order to ensure that $\mathbf{w}$ is an approximation of the conservative variables $\mathbf{u}$, and $\mathbf{z}$ an approximation of the flux $\mathbf{f}(\mathbf{u})$:

$$(\mathbf{w}, \mathbf{z}) \simeq (\mathbf{u}, \mathbf{f}(\mathbf{u})),$$

the kinetic steps are interlaced every time step Δt with relaxation steps, where $\mathbf{w}$ is kept unchanged and $\mathbf{z}$ is updated by

$$\mathbf{z} \leftarrow \theta \mathbf{f}(\mathbf{u}) + (1-\theta)\mathbf{z}. \tag{4}$$

When $\theta = 1$, one recovers the standard Jin-Xin relaxation algorithm. The over-relaxation case $\theta > 1$ is frequently used in applications in order to obtain better precision or additional properties. For instance, $\theta = 2$ ensures that $\mathbf{w} = \mathbf{u} + O(\Delta t^2)$.

2 Equivalent equation and boundary conditions

The objective of the talk is first to present how to obtain an equivalent PDE system to (2)-(3)-(4) when $\Delta t \to 0$. When $\theta = 2$, we can prove that, up to second order terms, $\mathbf{w}$ and $\mathbf{y} := \mathbf{z} - \mathbf{f}(\mathbf{w})$ satisfy

$$\partial_t \mathbf{w} + \partial_x \mathbf{f}(\mathbf{w}) = 0,$$

$$\partial_t \mathbf{y} - \mathbf{f}'(\mathbf{w}) \partial_x \mathbf{y} = 0.$$

This analysis is very useful for devising coherent boundary conditions to be applied to the additional variables $\mathbf{z}$ or $\mathbf{y}$.

In the second part of the talk we will present a way to apply boundary conditions with second order accuracy. Various numerical results and applications will also be presented. Some of the results are extracted from [1, 2].

References

[1] David Coulette, Emmanuel Franck, Philippe Helluy, Michel Mehrenberger, and Laurent Navoret. Palindromic discontinuous galerkin method for kinetic equations with stiff relaxation. *arXiv preprint arXiv:1612.09422*, 2016.

[2] Florence Drui, Emmanuel Franck, Philippe Helluy, and Laurent Navoret. An analysis of over-relaxation in a kinetic approximation of systems of conservation laws. *Comptes Rendus Mécanique*, 347(3):259–269, 2019.

[3] Shi Jin and Zhouping Xin. The relaxation schemes for systems of conservation laws in arbitrary space dimensions. *Communications on Pure and Applied Mathematics*, 48(3):235–276, 1995.

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