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FÜR PLASMAPHYSIK

A structure-preserving spline Finite-Element solver for the cold-plasma model

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EUROfusion

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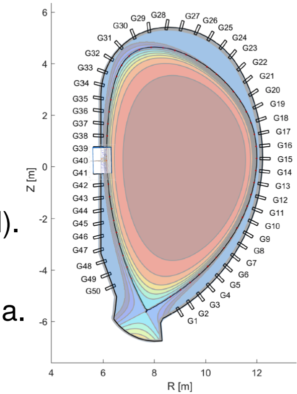
Motivation

- High-frequency waves in plasma.
- Applications in plasma-wave heating[1] and reflectometry diagnostics[2].



Problem setup:

- Strongly magnetized plasma (time-independent magnetic field).
- Time-independent electron plasma density.
- Time-harmonic electromagnetic wave launched into the plasma.

We run simulations using experimental data:

- SymPDE to set up model
- PSYDAC to simulate the time-evolution of the electromagnetic field

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[1] S. Heuraux et al.,"Simulation as a tool to improve wave heating in fusion plasmas", Journal of Plasma Physics 81,2015.

[2] F. da Silva et al., "An overview of the evolution of the modeling of reflectometry diagnostics in fusion plasmas using finite-difference time-domain codes", Fusion Engineering and Design 202, 2024.

[3] Image from F. da Silva et al. 2019 *JINST* **14** C08003.

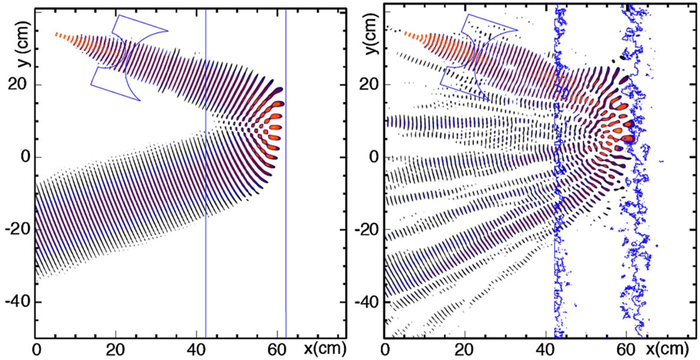
[4] https://github.com/pyccel/sympde

[5] https://github.com/pyccel/psydac

State of the art and challenge

Numerical modeling is difficult:

1. high space resolution,
2. many time-steps,
3. complex geometries,
4. complex wave-plasma interactions



Often solved in Frequency-Domain or Time-Domain (richer physics) but

- Finite-Difference on Cartesian grids,
- low order and long-time stability is an issue[3].

$\rightarrow$ Our discretization is high-order and structure-preserving, based on B-spline FEEC

[3] F. da Silva, M. Campos-Pinto, B. Després, S. Heuraux, "Stable explicit coupling of the Yee scheme with a linear current model in fluctuating magnetized plasmas", Journal of Computational Physics 295, 2015.

The cold-plasma model

- in time-domain
- Silver-Müller (aka impedance or Robin) boundary conditions

Time-domain problem

In normalized units (wrt. incoming wave frequency ω_0), the time evolution of the electromagnetic field $(\boldsymbol{E}, \boldsymbol{B})$ and the current density $\boldsymbol{J}$ is given by

$$\begin{aligned} \partial_t \boldsymbol{E} - \mathrm{curl} \boldsymbol{B} &= -\hat{\omega}_p \boldsymbol{Y}, \\ \partial_t \boldsymbol{B} + \mathrm{curl} \boldsymbol{E} &= \boldsymbol{0}, \\ \partial_t \boldsymbol{Y} + \hat{\omega}_c \boldsymbol{Y} \times \boldsymbol{b}_0 &= \hat{\omega}_p \boldsymbol{E} - \hat{\nu}_e \boldsymbol{Y}, \end{aligned} \tag{1}$$

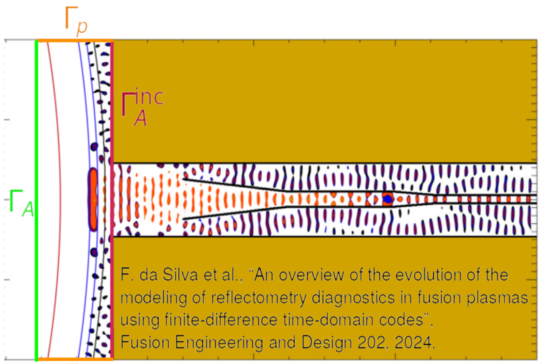
where $\boldsymbol{Y} = \omega_0 \hat{\omega}_p / 4\pi \boldsymbol{J}$, the normalized electron plasma and cyclotron frequencies are

$$\hat{\omega}_p(\boldsymbol{x}) = \sqrt{4\pi e^2 n_e(\boldsymbol{x})/(m_e \omega_0^2)}, \qquad \hat{\omega}_c(\boldsymbol{x}) = e|\boldsymbol{B}_0(\boldsymbol{x})|/(m_e c \omega_0),$$

- $n_e(\boldsymbol{x})$ is the background electron plasma density,
- $\boldsymbol{B}_0(\boldsymbol{x})$ is the background magnetic field with $\boldsymbol{b}_0 = \boldsymbol{B}_0/|\boldsymbol{B}_0|$,
- e is the elementary charge, m_e is the electron mass
- $\hat{\nu}_e \geq 0$ is the normalized electron-collision frequency.

Boundary conditions

- Instead of simulating the antenna, we prescribe a field on the boundary.
- The waves can leave the domain freely



Consider a partition $\partial\Omega = \Gamma_A \cup \Gamma_p$, let $\boldsymbol{\nu}$ be the outward unit normal,

- Periodic boundary conditions on Γ_p,
- Silver-Müller boundary conditions on Γ_A:

$$\boldsymbol{\nu} \times (\boldsymbol{E} - \boldsymbol{B} \times \boldsymbol{\nu}) = \begin{cases} \boldsymbol{\nu} \times \boldsymbol{s}^{\text{inc}}(\boldsymbol{x}, t) & \text{on } \Gamma_A^{\text{inc}}, \\ 0 & \text{on } \Gamma_A \backslash \Gamma_A^{\text{inc}}, \end{cases} \tag{2}$$

Weak form

- How to solve the system with the Silver-Müller boundary conditions (SM BC)?

$$
\begin{aligned}
\partial_t \boldsymbol{E} - \mathrm{curl} \boldsymbol{B} &= -\hat{\omega}_p \boldsymbol{Y} && \text{in } \Omega, \\
\partial_t \boldsymbol{B} + \mathrm{curl} \boldsymbol{E} &= \boldsymbol{0} && \text{in } \Omega, \\
\partial_t \boldsymbol{Y} + \hat{\omega}_c \boldsymbol{Y} \times \boldsymbol{b}_0 &= \hat{\omega}_p \boldsymbol{E} - \hat{\nu}_e \boldsymbol{Y} && \text{in } \Omega, \\
\boldsymbol{\nu} \times (\boldsymbol{E} - \boldsymbol{B} \times \boldsymbol{\nu}) &= \boldsymbol{\nu} \times \boldsymbol{s}^{\mathrm{inc}}(\boldsymbol{x}, t) && \text{on } \Gamma_A.
\end{aligned}
$$

Weak form

- How to solve the system with the Silver-Müller boundary conditions (SM BC)?
  weak form $=$ system of equations that solves the system with SM BC.
  - Test against a test function and integrate over domain.
  - Integration by parts $\rightarrow$ use boundary integral to enforce boundary conditions.
  - Choose the right spaces for every field.

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- We use the "strong-Faraday and weak-Ampère" formulation.

Find $(\boldsymbol{E}, \boldsymbol{B}, \boldsymbol{Y}) \in C^1(\mathbb{R}_+; V)$ such that for every $(\boldsymbol{F}, \boldsymbol{C}, \boldsymbol{G}) \in V$,

$$\begin{aligned}
\langle \boldsymbol{F}, \partial_t \boldsymbol{E}\rangle_\Omega - \langle \mathrm{curl}\boldsymbol{F}, \boldsymbol{B}\rangle_\Omega + \langle \boldsymbol{\nu} \times \boldsymbol{F}, \boldsymbol{\nu} \times \boldsymbol{E}\rangle_{\Gamma_A} + \langle \boldsymbol{F}, \hat{\omega}_p \boldsymbol{Y}\rangle_\Omega =& \langle \boldsymbol{\nu} \times \boldsymbol{F}, \boldsymbol{\nu} \times \boldsymbol{s}^{\mathrm{inc}}\rangle_{\Gamma_A^{\mathrm{inc}}}, && \text{(weak)}\\
\langle \boldsymbol{C}, \partial_t \boldsymbol{B}\rangle_\Omega + \langle \boldsymbol{C}, \mathrm{curl}\boldsymbol{E}\rangle_\Omega =& 0, && \text{(strong)}\\
\langle \boldsymbol{G}, \partial_t \boldsymbol{Y}\rangle_\Omega + \langle \boldsymbol{G} \times \boldsymbol{Y}, \hat{\omega}_c \boldsymbol{b}_0\rangle_\Omega - \langle \boldsymbol{G}, \hat{\omega}_p \boldsymbol{E}\rangle_\Omega + \langle \boldsymbol{G}, \hat{\nu}_e \boldsymbol{Y}\rangle_\Omega =& 0, && \text{(strong)}
\end{aligned}$$

where $\langle \boldsymbol{U}, \boldsymbol{W}\rangle_\Omega := \int_\Omega \boldsymbol{U} \cdot \boldsymbol{W} d\boldsymbol{x}$.

Weak form

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\langle \boldsymbol{C}, \partial_t \boldsymbol{B}\rangle_\Omega + \langle \boldsymbol{C}, \mathrm{curl}\boldsymbol{E}\rangle_\Omega &= 0, && \text{(strong)}\\
\langle \boldsymbol{G}, \partial_t \boldsymbol{Y}\rangle_\Omega + \langle \boldsymbol{G} \times \boldsymbol{Y}, \hat{\omega}_c \boldsymbol{b}_0\rangle_\Omega - \langle \boldsymbol{G}, \hat{\omega}_p \boldsymbol{E}\rangle_\Omega + \langle \boldsymbol{G}, \hat{\nu}_e \boldsymbol{Y}\rangle_\Omega &= 0. && \text{(strong)}
\end{aligned}
$$

What is V? Choose the right spaces:

$$\boldsymbol{E} \in H^{\mathrm{curl}}(\Omega), \qquad \boldsymbol{B} \in H^{\mathrm{div}}(\Omega), \qquad \boldsymbol{Y} \in H^{\mathrm{curl}}(\Omega), \qquad \boldsymbol{\nu} \times \boldsymbol{E} \in (L^2(\partial\Omega))^3$$

Structure of the time-domain problem

$$V := \{(\boldsymbol{E}, \boldsymbol{B}, \boldsymbol{Y}) \in H^{\mathrm{curl}}(\Omega) \times H^{\mathrm{div}}(\Omega) \times H^{\mathrm{curl}}(\Omega) | \boldsymbol{\nu} \times \boldsymbol{E} \in (L^2(\partial\Omega))^3\}.$$

- Find $(\boldsymbol{E}, \boldsymbol{B}, \boldsymbol{Y}) \in C^1(\mathbb{R}_+; V)$ such that for every $(\boldsymbol{F}, \boldsymbol{C}, \boldsymbol{G}) \in V$,

$$\begin{aligned}\langle \boldsymbol{F}, \partial_t \boldsymbol{E}\rangle_\Omega - \langle \mathrm{curl}\boldsymbol{F}, \boldsymbol{B}\rangle_\Omega + \langle \boldsymbol{\nu} \times \boldsymbol{F}, \boldsymbol{\nu} \times \boldsymbol{E}\rangle_{\Gamma_A} + \langle \boldsymbol{F}, \hat{\omega}_p \boldsymbol{Y}\rangle_\Omega &= \langle \boldsymbol{\nu} \times \boldsymbol{F}, \boldsymbol{\nu} \times \boldsymbol{s}^{\mathrm{inc}}\rangle_{\Gamma_A^{\mathrm{inc}}}, \\ \langle \boldsymbol{C}, \partial_t \boldsymbol{B}\rangle_\Omega + \langle \boldsymbol{C}, \mathrm{curl}\boldsymbol{E}\rangle_\Omega &= 0, \\ \langle \boldsymbol{G}, \partial_t \boldsymbol{Y}\rangle_\Omega + \langle \boldsymbol{G} \times \boldsymbol{Y}, \hat{\omega}_c \boldsymbol{b}_0\rangle_\Omega - \langle \boldsymbol{G}, \hat{\omega}_p \boldsymbol{E}\rangle_\Omega + \langle \boldsymbol{G}, \hat{\nu}_e \boldsymbol{Y}\rangle_\Omega &= 0,\end{aligned}$$

- Non-Hamiltonian terms spoil energy conservation:
  boundary conditions and dissipative collisions

- Energy-balance

$$\partial_t \mathcal{H} = -\|\boldsymbol{\nu} \times \boldsymbol{E}\|^2_{L^2(\Gamma_A)} - \langle \hat{\nu}_e \boldsymbol{Y}, \boldsymbol{Y}\rangle_\Omega + \langle \boldsymbol{\nu} \times \boldsymbol{E}, \boldsymbol{\nu} \times \boldsymbol{s}^{\mathrm{inc}}\rangle_{\Gamma_A^{\mathrm{inc}}},$$

  where $\mathcal{H} = \frac{1}{2}\|\boldsymbol{U}\|^2 = \frac{1}{2}\left(\|\boldsymbol{E}\|^2_{L^2(\Omega)} + \|\boldsymbol{B}\|^2_{L^2(\Omega)} + \|\boldsymbol{Y}\|^2_{L^2(\Omega)}\right)$

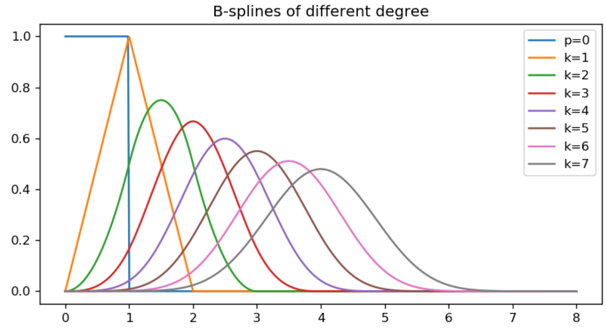
- Long-time stability

$$\|(\boldsymbol{U} - \boldsymbol{U}^{\mathrm{th}})(t)\| \leq \|(\boldsymbol{U} - \boldsymbol{U}^{\mathrm{th}})(0)\| \qquad \forall t > 0,$$

  where $\boldsymbol{U}^{\mathrm{th}}$ is the time-harmonic solution.

Structure-preserving discretization

- Space discretization: B-splines Finite Element Exterior Calculus (FEEC)
  - $\mathrm{div}\boldsymbol{B} = 0$ guaranteed for all time
  - easily applicable to **curvilinear / multipatch** domains
  - high-order: choice of **B-spline degree**
- Time discretization: time-splitting methods (Poisson and Hamiltonian splitting)
  - **preservation** properties: energy, total charge, etc.
  - capture time-domain richer physics: **energy-balance** given by trade-off of boundary conditions and electron-collision dissipation
  - inherit **long-time stability**
  - **lower cost**: solve subsystems instead of full system
  - high-order: choice of subsystem **integrator and composition**.
- Result: **high-order** discretization that **preserves physical behaviour.**
  - interest for **long-time** simulations
  - compatible decompositions: **incoming/scattered**, **time-harmonic/transient** fields



Space discretization

- **B-splines $\longrightarrow$ high order, CAD geometries**
- **FEEC (Finite Element Exterior Calculus) $\longrightarrow$ structure preservation**

Idea of B-splines FEEC framework

- Work with arrays (B-splines coefficients) and matrices (operators)

- Discrete differential operators are **exact** for fields in a certain basis (*Finite Element (FE) fields*)

- The only approximation is the projection to FE fields (approximation error adjusted via B-spline degree).

- Structure-preservation by construction, e.g.

$$\mathrm{div}\mathrm{curl}\boldsymbol{E} = 0 \qquad \mathrm{curl}\mathrm{grad}V_{\boldsymbol{E}} = 0$$

- Example of use:

to compute $\mathrm{curl}\boldsymbol{F}$, we compute the matrix-vector product $\mathbb{C}\mathbf{F}$,

where $\mathbf{F}$ are the B-spline coefficients and $\mathbb{C}$ is the curl matrix .

FEEC framework

Fields $\boldsymbol{F}(\boldsymbol{x})$

$\downarrow$

FE fields $\sum \mathrm{F}_i \Lambda_i^k(\boldsymbol{x})$

$\downarrow$

Coefficients F_i

$$
\begin{tikzcd}
H^1(\Omega) \arrow[r, "\mathrm{grad}"] \arrow[d, "\Pi_0"] & H^{\mathrm{curl}}(\Omega) \arrow[r, "\mathrm{curl}"] \arrow[d, "\Pi_1"] & H^{\mathrm{div}}(\Omega) \arrow[r, "\mathrm{div}"] \arrow[d, "\Pi_2"] & L^2(\Omega) \arrow[d, "\Pi_3"] \\
V_h^0 \arrow[r, "\mathrm{grad}"] \arrow[d, "\boldsymbol{\sigma}^0"] & V_h^1 \arrow[r, "\mathrm{curl}"] \arrow[d, "\boldsymbol{\sigma}^1"] & V_h^2 \arrow[r, "\mathrm{div}"] \arrow[d, "\boldsymbol{\sigma}^2"] & V_h^3 \arrow[d, "\boldsymbol{\sigma}^3"] \\
C_h^0 \arrow[r, "\mathbb{G}"] & C_h^1 \arrow[r, "\mathbb{C}"] & C_h^2 \arrow[r, "\mathbb{D}"] & C_h^3
\end{tikzcd}
\tag{3}
$$

- $V_h^k = \mathrm{span}_i \left\{\Lambda_i^k\right\}$ are the discrete spaces (tensor-product 1D B-splines degree p),
- Π_k are the commuting projections and $\boldsymbol{\sigma}^k$, the discrete degrees of freedom.
- $\mathbb{G}, \mathbb{C}, \mathbb{D}$ are the gradient, curl and divergence matrices.
- for $\boldsymbol{F}_h \in V_h^1$, the field $\mathrm{curl}\boldsymbol{F}_h$ has coefficients $\mathbb{C}\mathbf{F}$

  $\rightarrow$ discrete differential operators are *exact*.

Semi-discrete form

Space discretization yields ODE in terms of coefficients $(\mathbf{E}, \mathbf{B}, \mathbf{Y})(t) \in C_h^1 \times C_h^2 \times C_h^1$

$$\begin{bmatrix} \partial_t \mathbf{E} \\ \partial_t \mathbf{B} \\ \partial_t \mathbf{Y} \end{bmatrix} = \begin{bmatrix} -\mathbb{M}_1^{-1}\mathbb{A}_1 & \mathbb{M}_1^{-1}\mathbb{C}^T\mathbb{M}_2 & -\mathbb{M}_1^{-1}\mathbb{M}_{1,\hat{\omega}_p} \\ -\mathbb{C} & \mathbb{0} & \mathbb{0} \\ \mathbb{M}_1^{-1}\mathbb{M}_{1,\hat{\omega}_p} & \mathbb{0} & -\mathbb{M}_1^{-1}\left(\mathbb{R}_{1,\hat{\omega}_c} + \mathbb{M}_{1,\hat{\nu}_e}\right) \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{B} \\ \mathbf{Y} \end{bmatrix} + \begin{bmatrix} \mathbb{M}_1^{-1}\mathbf{S}^{\mathrm{inc}}(t) \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \quad (4)$$

where $\mathbb{C}$ is the curl matrix, $\mathbb{C}^T$ is its transpose,

$$(\mathbb{M}_1)_{i,j} := \langle \mathbf{\Lambda}_i^1, \mathbf{\Lambda}_j^1 \rangle_\Omega, \qquad (\mathbb{M}_2)_{i,j} := \langle \mathbf{\Lambda}_i^2, \mathbf{\Lambda}_j^2 \rangle_\Omega,$$

$$(\mathbb{A}_1)_{i,j} := \langle \boldsymbol{\nu} \times \mathbf{\Lambda}_i^1, \boldsymbol{\nu} \times \mathbf{\Lambda}_j^1 \rangle_{\Gamma_A}, \qquad (\mathbb{M}_{1,\hat{\omega}_p})_{i,j} := \langle \mathbf{\Lambda}_i^1, \hat{\omega}_p \mathbf{\Lambda}_j^1 \rangle_\Omega,$$

$$(\mathbb{M}_{1,\hat{\nu}_e})_{i,j} := \langle \mathbf{\Lambda}_i^1, \hat{\nu}_e \mathbf{\Lambda}_j^1 \rangle_\Omega, \qquad (\mathbb{R}_{1,\hat{\omega}_c})_{i,j} := \langle \mathbf{\Lambda}_i^1 \times \mathbf{\Lambda}_j^1, \hat{\omega}_c \boldsymbol{b}_0 \rangle_\Omega,$$

and the incoming source is given by

$$(\mathbf{S}^{\mathrm{inc}}(t))_i = \langle \boldsymbol{\nu} \times \mathbf{\Lambda}_i^1, \boldsymbol{\nu} \times \boldsymbol{s}^{\mathrm{inc}}(t) \rangle_{\Gamma_A^{\mathrm{inc}}} = \langle \boldsymbol{\nu} \times \mathbf{\Lambda}_i^1, \boldsymbol{\nu} \times \mathrm{Re}\{\hat{\boldsymbol{s}}^{\mathrm{inc}} e^{-\mathrm{i}t}\} \rangle_{\Gamma_A^{\mathrm{inc}}}.$$

$\mathbb{M}_1$ has a Kronecker structure $\rightarrow \mathbb{M}_1^{-1}$ is cheap to compute.

Structure of the semi-discrete form

Let $\mathrm{H} = \mathcal{H}(\boldsymbol{E}_h, \boldsymbol{B}_h, \boldsymbol{Y}_h)$ be the discrete Hamiltonian, $\mathbf{U} = (\mathbf{E}, \mathbf{B}, \mathbf{Y})$.

- Hamiltonian systems have the form $\partial_t \mathbf{U} = \mathbb{P}\nabla_{\mathbf{U}}\mathrm{H}$ with $\mathbb{P}$ antisymmetric.
- Here we have dissipation and boundary terms, i.e.

$$\partial_t \mathbf{U} = \mathbb{P}\nabla_{\mathbf{U}}\mathrm{H} + \mathbb{N}\nabla_{\mathbf{U}}\mathrm{H} + \mathbf{f}(t),$$

where the Poisson and "metric" matrices are

$$\mathbb{P} := \begin{bmatrix} \mathbb{0} & \mathbb{M}_1^{-1}\mathbb{C}^T & -\mathbb{M}_1^{-1}\mathbb{M}_{1,\hat{\omega}_p}\mathbb{M}_1^{-1} \\ -\mathbb{C}\mathbb{M}_1^{-1} & \mathbb{0} & \mathbb{0} \\ \mathbb{M}_1^{-1}\mathbb{M}_{1,\hat{\omega}_p}\mathbb{M}_1^{-1} & \mathbb{0} & -\mathbb{M}_1^{-1}\mathbb{R}_{1,\hat{\omega}_c}\mathbb{M}_1^{-1} \end{bmatrix}, \; \mathbb{N} := \begin{bmatrix} -\mathbb{M}_1^{-1}\mathbb{A}_1\mathbb{M}_1^{-1} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & -\mathbb{M}_1^{-1}\mathbb{M}_{1,\hat{\nu}_e}\mathbb{M}_1^{-1} \end{bmatrix}$$

and $\mathbf{f}(t) := [\mathbb{M}_1^{-1}\mathbf{S}^{\mathrm{inc}}(t), \mathbf{0}, \mathbf{0}]^T$.

- Energy-balance

$$\partial_t \mathrm{H} = -(\mathbf{E}\cdot\mathbb{A}_1\mathbf{E} + \mathbf{Y}\cdot\mathbb{M}_{1,\hat{\nu}_e}\mathbf{Y}) + \mathbf{E}\cdot\mathbf{S}^{\mathrm{inc}}(t)$$

- Long-time stability

$$\|(\boldsymbol{U}_h - \boldsymbol{U}^{\mathrm{th}})(t)\| \leq \|(\boldsymbol{U}_h - \boldsymbol{U}^{\mathrm{th}})(0)\| \qquad \forall t > 0$$

$$\begin{bmatrix} \partial_t \mathbf{E} \\ \partial_t \mathbf{B} \\ \partial_t \mathbf{Y} \end{bmatrix} = \begin{bmatrix} -\mathbb{M}_1^{-1}\mathbb{A}_1 & \mathbb{M}_1^{-1}\mathbb{C}^T\mathbb{M}_2 & -\mathbb{M}_1^{-1}\mathbb{M}_{1,\hat{\omega}_p} \\ -\mathbb{C} & \mathbb{0} & \mathbb{0} \\ \mathbb{M}_1^{-1}\mathbb{M}_{1,\hat{\omega}_p} & \mathbb{0} & -\mathbb{M}_1^{-1}\left(\mathbb{R}_{1,\hat{\omega}_c} + \mathbb{M}_{1,\hat{\nu}_e}\right) \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{B} \\ \mathbf{Y} \end{bmatrix} + \begin{bmatrix} \mathbb{M}_1^{-1}\mathbf{S}^{\text{inc}}(t) \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

Time discretization

$$\partial_t \mathbf{U} = \mathbb{P}\nabla_{\mathbf{U}}\mathsf{H} + \mathbb{N}\nabla_{\mathbf{U}}\mathsf{H} + \mathbf{f}(t)$$

- **Use splitting methods for Hamiltonian systems**
- **Handle non-Hamiltonian terms**

Poisson splitting

- For the Hamiltonian part, split Poisson matrix:

$$\mathbb{P}_{\text{Maxwell}} = \begin{bmatrix} \mathbb{0} & \mathbb{M}_1^{-1}\mathbb{C}^T & \mathbb{0} \\ -\mathbb{C}\mathbb{M}_1^{-1} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{0} \end{bmatrix}, \quad \mathbb{P}_{\text{plasma}} = \begin{bmatrix} \mathbb{0} & \mathbb{0} & -\mathbb{M}_1^{-1}\mathbb{M}_{1,\hat{\omega}_p}\mathbb{M}_1^{-1} \\ \mathbb{0} & \mathbb{0} & \mathbb{0} \\ \mathbb{M}_1^{-1}\mathbb{M}_{1,\hat{\omega}_p}\mathbb{M}_1^{-1} & \mathbb{0} & -\mathbb{M}_1^{-1}\mathbb{R}_{1,\hat{\omega}_c}\mathbb{M}_1^{-1} \end{bmatrix},$$

- Keep boundary terms with "$\mathrm{curl}\boldsymbol{B}$" term (do not split integration by parts, enforce physical boundary conditions).

$$(\mathbb{P}_{\text{Maxwell}}) \begin{cases} \mathbb{M}_1 \partial_t \mathbf{E} - \mathbb{C}^T \mathbb{M}_2 \mathbf{B} + \mathbb{A}_1 \mathbf{E} = \mathbf{S}^{\text{inc}}(t) \\ \partial_t \mathbf{B} + \mathbb{C}\mathbf{E} = \mathbf{0} \\ \partial_t \mathbf{Y} = \mathbf{0} \end{cases} \qquad (\mathbb{P}_{\text{plasma}}) \begin{cases} \mathbb{M}_1 \partial_t \mathbf{E} + \mathbb{M}_{1,\hat{\omega}_p} \mathbf{Y} = \mathbf{0} \\ \partial_t \mathbf{B} = \mathbf{0} \\ \mathbb{M}_1 \partial_t \mathbf{Y} - \mathbb{M}_{1,\hat{\omega}_p} \mathbf{E} + \mathbb{R}_{1,\hat{\omega}_c} \mathbf{Y} + \mathbb{M}_{1,\hat{\nu}_e} \mathbf{Y} = \mathbf{0} \end{cases}$$

- Long-time stability, (no CFL condition):

$$\|(\boldsymbol{U}_h - \boldsymbol{U}_h^{\text{th}})(n\Delta t)\| \leq \|(\boldsymbol{U}_h - \boldsymbol{U}_h^{\text{th}})(0)\| \qquad \forall n \geq 0.$$

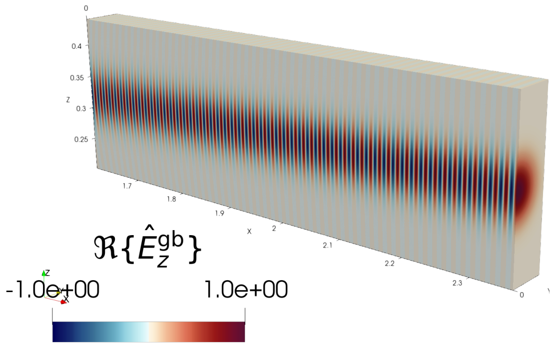
Results

i. **Heating setup experiment (high-frequency)**
ii. **Reflectometry experiment**

Heating setup experiment

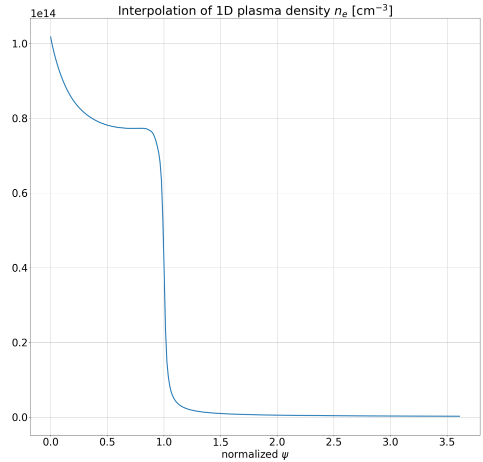
- Domain is $[160, 236] \times [-6, 6] \times [20, 44]$ (in cm).
- Incoming wave is a Gaussian beam with $f_0 = 140$ GHz, beam width is 3.5 cm, launching point is $(236, 0, 32)$, polarized in $\hat{\boldsymbol{z}}$.
- Parameters taken from ASDEX Upgrade shot #25485 [4]
- Heating setup: second harmonic resonance ($\hat{\omega}_c = 0.5$) at $x = 165$ cm.

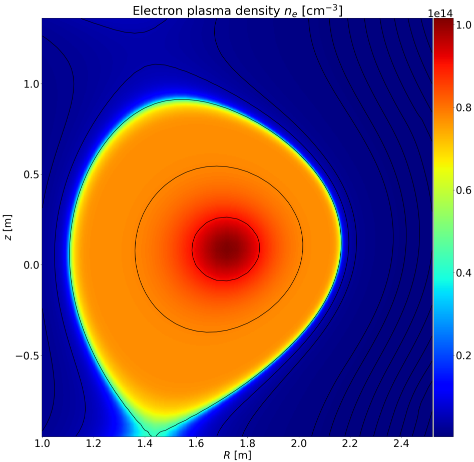
Launched beam (in vacuum)

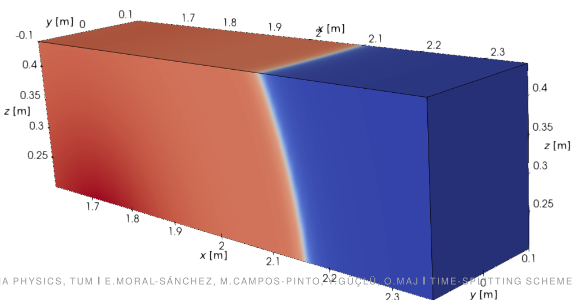


[4] Data and setup provided by O. Maj.

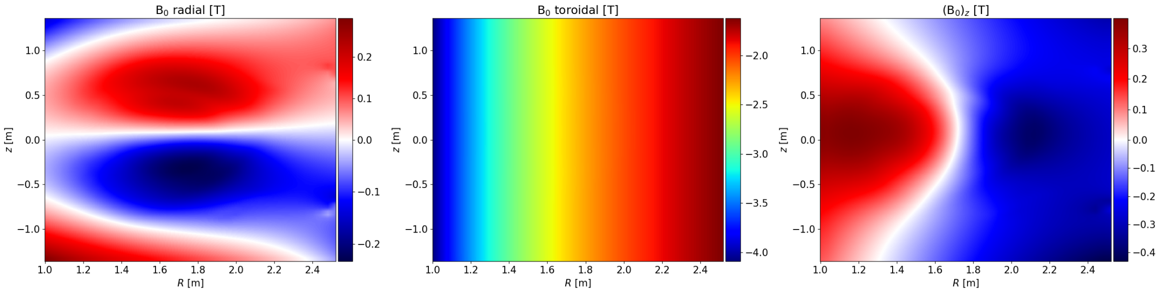
Heating experiment: electron plasma density

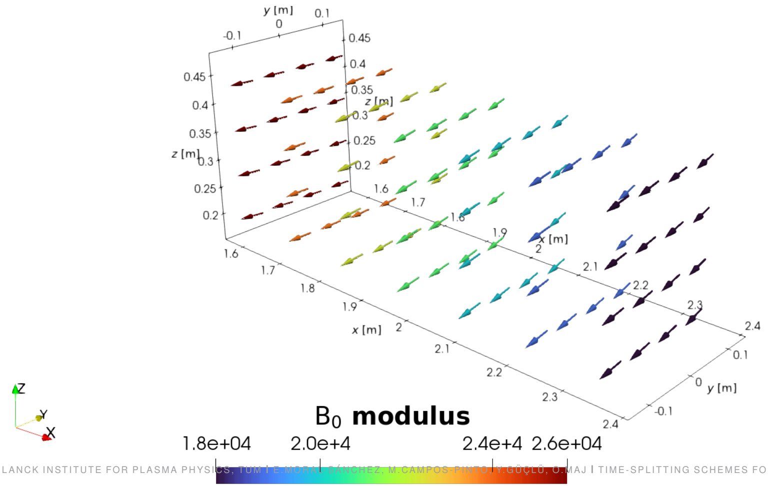






Heating experiment: background magnetic field

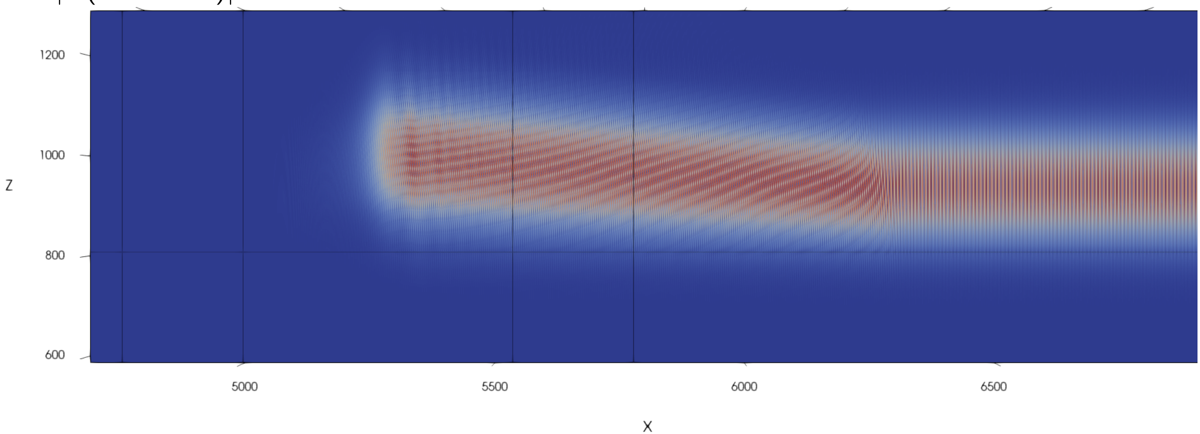




Electric field at $t = 300T$

- number of cells: $[3547, 1, 560]$ (10 cells per wavelength along x, 5 along z).
- $CFL = 0.21$
- B-spline degree: $[3, 1, 2]$
- periodic BC along y (force *constant* direction)

$|\boldsymbol{E}(t = 300T)|$

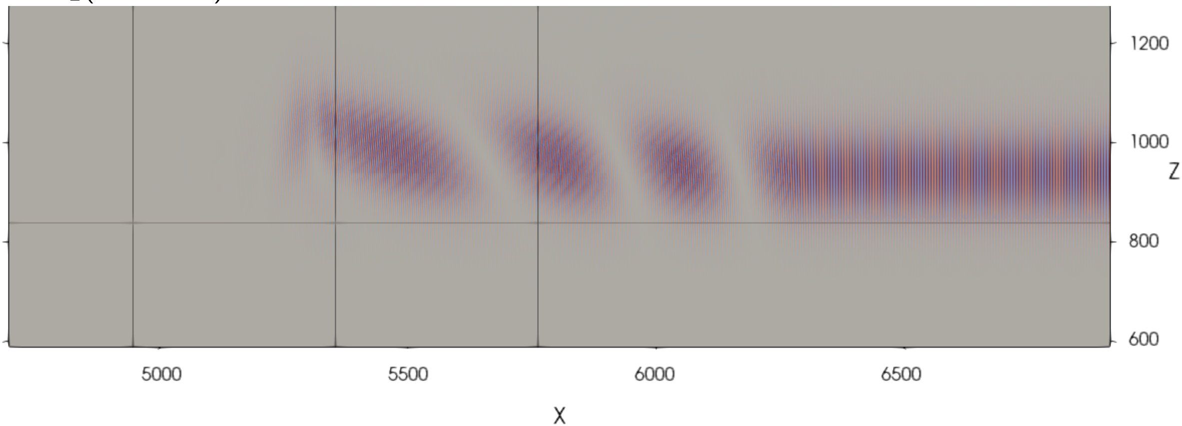


Ran in TOKb cluster, 32 cores, max RAM=1.2Tb (matrix assembly)

Electric field at $t = 300T$

- number of cells: $[3547, 1, 560]$ (10 cells per wavelength along x, 5 along z).
- $CFL = 0.21$
- B-spline degree: $[3, 1, 2]$
- periodic BC along y (force *constant* direction)

$E_z(t = 300T)$

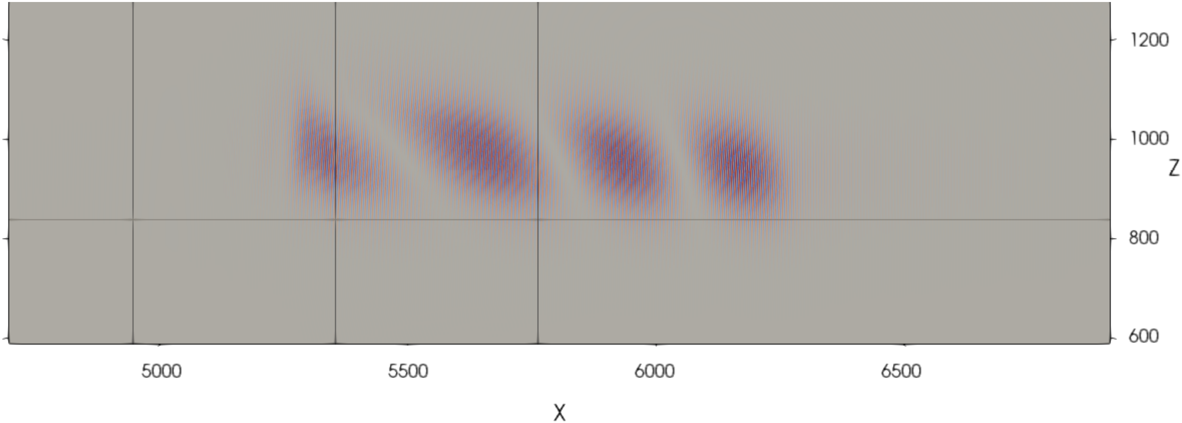


Ran in TOKb cluster, 32 cores, max RAM=1.2Tb (matrix assembly)

Electric field at $t = 300T$

- number of cells: $[3547, 1, 560]$ (10 cells per wavelength along x, 5 along z).
- $CFL = 0.21$
- B-spline degree: $[3, 1, 2]$
- periodic BC along y (force *constant* direction)

$E_y(t = 300T)$

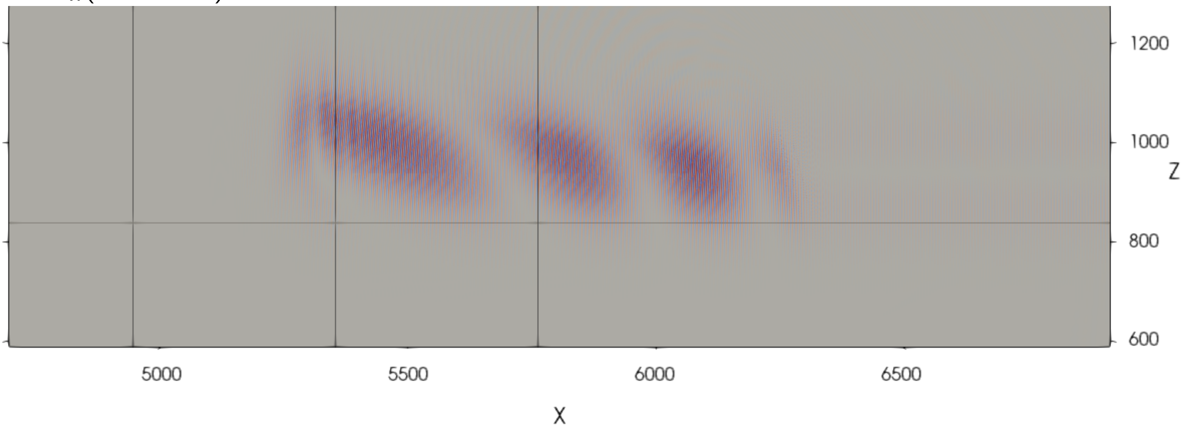


Ran in TOKb cluster, 32 cores, max RAM=1.2Tb (matrix assembly)

Electric field at $t = 300T$

- number of cells: $[3547, 1, 560]$ (10 cells per wavelength along x, 5 along z).
- $CFL = 0.21$
- B-spline degree: $[3, 1, 2]$
- periodic BC along y (force *constant* direction)

$E_x(t = 300T)$

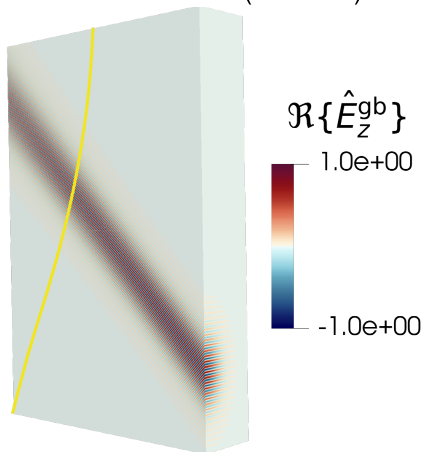


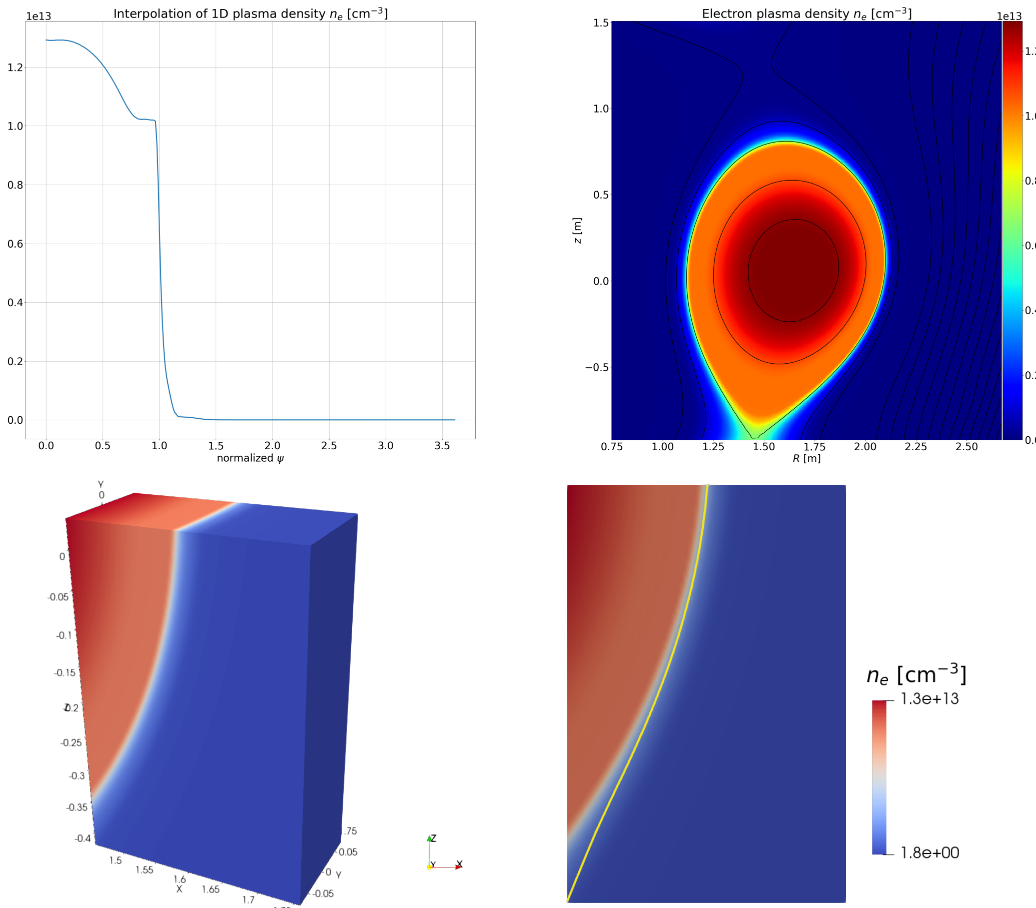
Ran in TOKb cluster, 32 cores, max RAM=1.2Tb (matrix assembly)

Reflectometry experiment

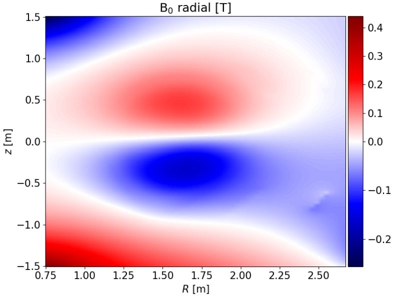
- Incoming wave is a Gaussian beam with $f_0 = 59$ GHz, launched with an angle of $\pi/4$ rad $= 45^\circ$ wrt. normal, beam width is 3 cm and polarization is $\hat{\boldsymbol{z}}$.
- Parameters taken from ASDEX Upgrade shot #30907

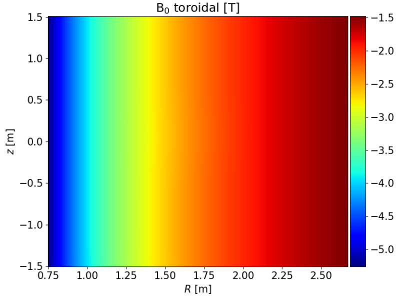
Launched beam (vacuum)

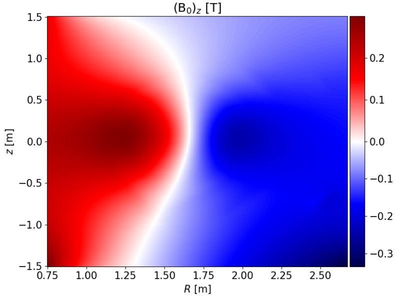


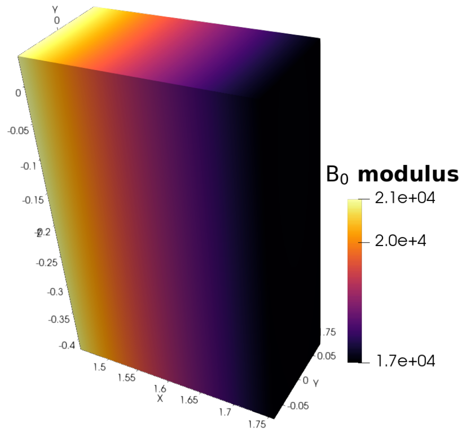

Interpolation of 1D plasma density n_e [cm^{-3}]
1e13
normalized ψ
Electron plasma density n_e [cm^{-3}]
1e13
z [m]
R [m]
X
Y
Z
n_e [cm^{-3}]
1.3e+13
1.8e+00

Reflectometry experiment: background magnetic field



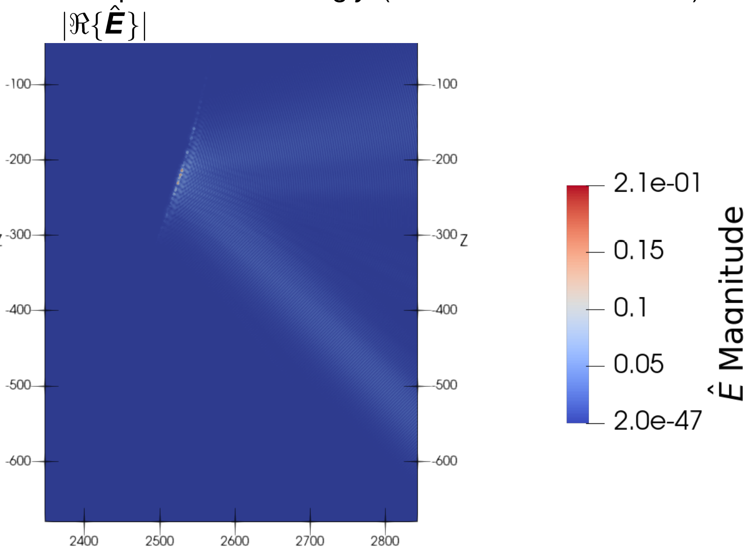


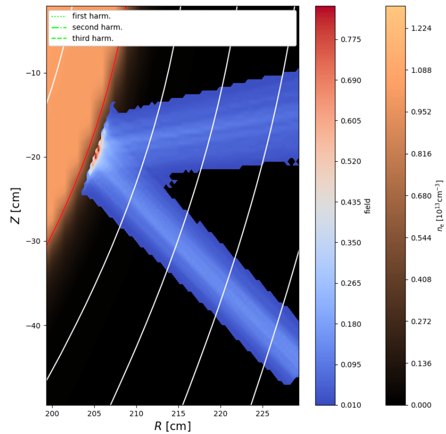




Time-harmonic electric field

- number of cells: $[280, 1, 400]$.
- solve in frequency-domain
- B-spline degree: $[3, 1, 3]$
- periodic BC along y (force *constant* direction)



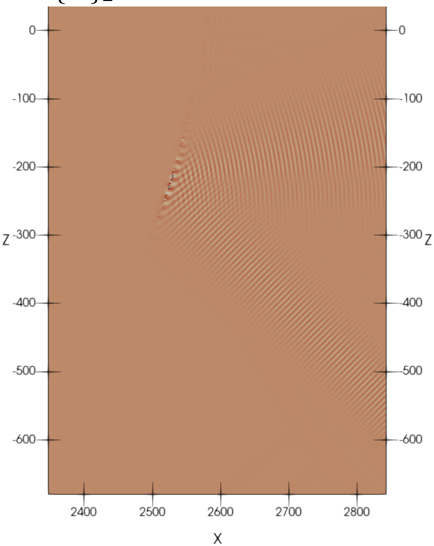


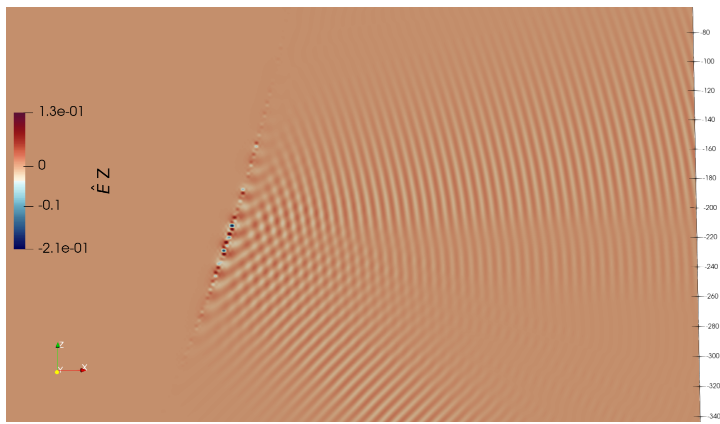
[4] Right figure: WKB beam simulation provided by O. Maj

Time-harmonic electric field

- number of cells: $[280, 1, 400]$.
- solve in frequency-domain
- B-spline degree: $[3, 1, 3]$
- periodic BC along y (force *constant* direction)

$\Re\{\hat{\boldsymbol{E}}\}_z$

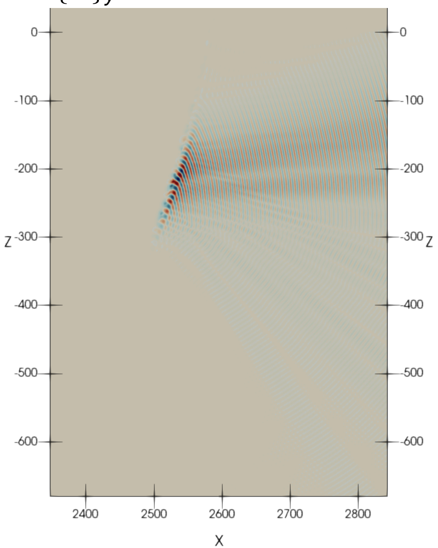


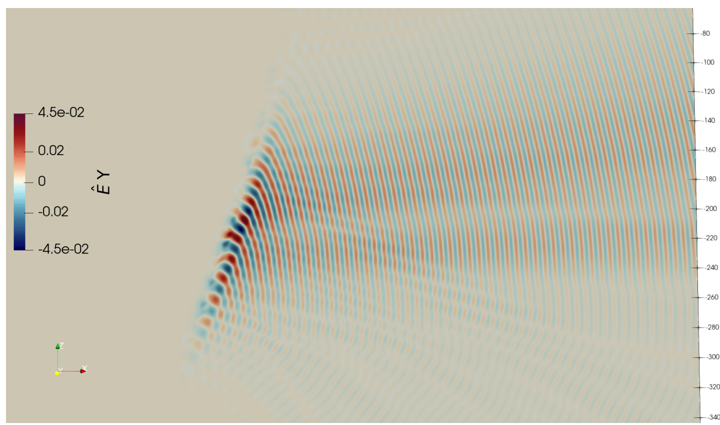


Time-harmonic electric field

- number of cells: $[280, 1, 400]$.
- solve in frequency-domain
- B-spline degree: $[3, 1, 3]$
- periodic BC along y (force *constant* direction)

$\Re\{\hat{\boldsymbol{E}}\}_y$

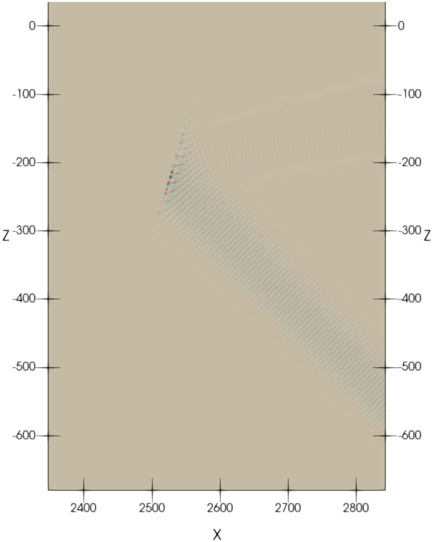


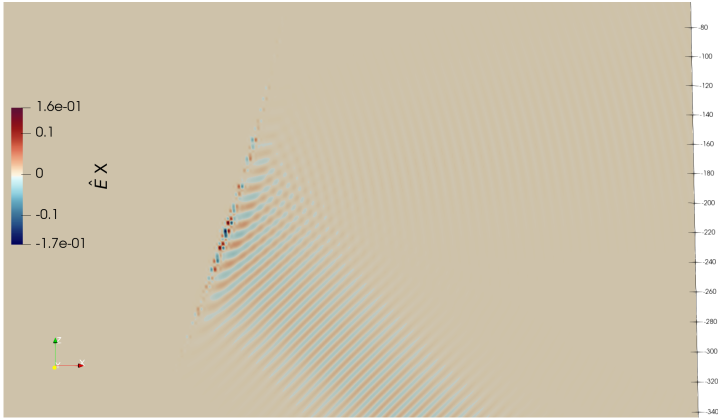


Time-harmonic electric field

- number of cells: $[280, 1, 400]$.
- solve in frequency-domain
- B-spline degree: $[3, 1, 3]$
- periodic BC along y (force *constant* direction)

$\Re\{\hat{\boldsymbol{E}}\}_x$

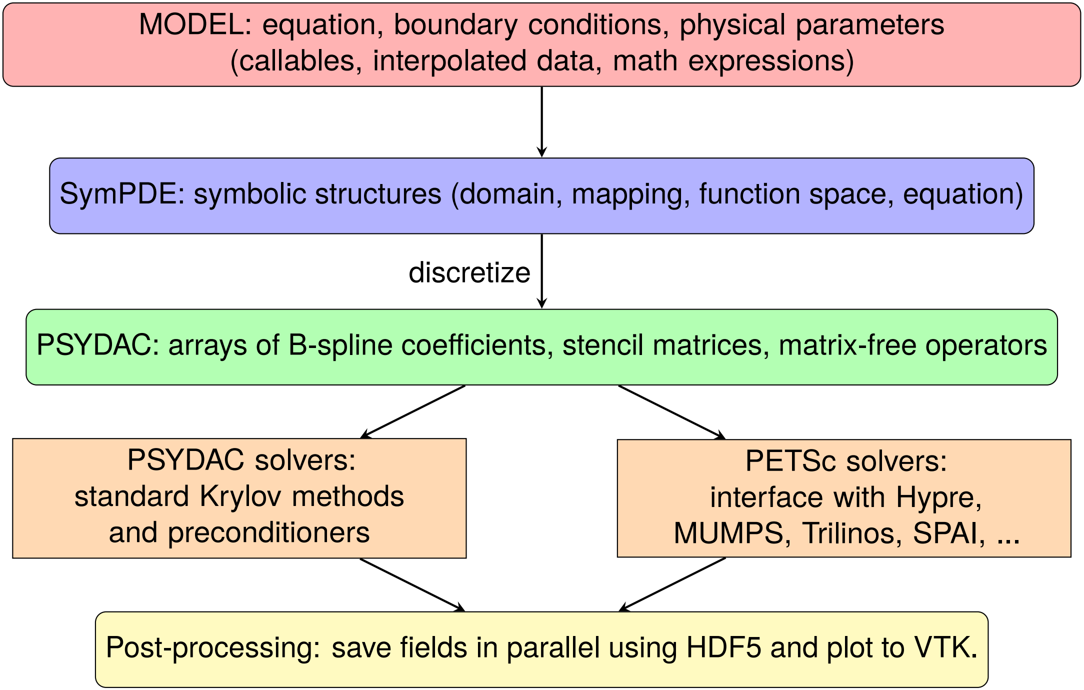




How to use Psydac

- **high-order structure-preserving space discretization (B-splines FEEC)**
- **set up the problem symbolically, internally work with arrays**
- **run in parallel using MPI, including post-processing**

How to run a simulation with Psydac[5]



[5] https://github.com/pyccel/psydac

Set up Poisson equation using Sympde

Let $\Omega = [0,5] \times [-1, 3.7] \times [0, 2\pi]$,

$$\Delta u = f \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } x = 0,\ x = 5,$$

and periodic along y, z.

Find $u \in H_0^1(\Omega)$ such that

$$\langle \nabla u, \nabla w \rangle_{L^2(\Omega)} = \langle f, w \rangle_{L^2(\Omega)}$$

for every $w \in H_0^1(\Omega)$.

f can be a callable (interpolation) or a symbolic expression.

```
# domain
D = Cube('D', bounds1=(0, 5),
              bounds2=(-1, 3.7),
              bounds3=(0, 2*np.pi))

# function space
V = ScalarFunctionSpace('V', D, kind='h1')
u = element_of(V, name='u') # trial function
w = element_of(V, name='w') # test function

# LHS
a = BilinearForm((u,v),
           integral(D, dot(grad(v), grad(u))))

# RHS
l = LinearForm(v , integral(D, f * v))

# homogeneous Dirichlet boundary conditions
bdry0 = Union(D.get_boundary(axis=0, ext=-1)
    ,
              D.get_boundary(axis=0, ext=1))
bc = EssentialBC(u, 0, bdry0)

# declare equation
equation = find(u, forall=w,
                lhs=a(u,w), rhs=l(w), bc=bc)
```

Discretize and solve using Psydac

Discretization ingredients:

- Number of cells
- B-spline degree
- Backend
- communicator

Solver configuration

- Type (CG)
- Tolerance
- maximum number of iterations

The result are the B-spline coefficients of the solution.

```
# discretize domain
Dh = discretize(D, ncells=(30,20,5),
                   periodic=(False,True,True),
                   comm=MPI.COMM_WORLD)

# discretize function space
Vh = discretize(V, Dh, degree=(3,1,2))

# discretize equation
be=PSYDAC_BACKENDS['pyccel-gcc']

equation_h = discretize(equation, Dh,
                        [Vh, Vh], backend=be)

# solve discrete equation
equation_h.set_solver('cg', tol=1e-9,
                      maxiter=100)
u_h = equation_h.solve() # coefficients array

# retrieve a callable field
u_callable = FemField(Vh, coeffs=u_h)
```

Conclusions

- B-splines FEEC provides a **high order structure-preserving space discretization**.

- Our time discretization is **high order**, captures the **time-domain richer physics** and is **efficient**.

- Psydac provides a framework for structure-preserving simulations.
  - Using SymPDE we can set up a problem with **complicated boundary conditions** and **general curvilinear / multipatch geometries**.
  - Psydac discretizes the symbolic structures, provides different solvers and powerful post-processing tools.

For more details on the schemes, see E. Moral Sánchez, M. Campos Pinto, Y. Güçlü and O. Maj ,"Time-splitting methods for the cold-plasma model using Finite Element Exterior Calculus", arXiv:2501.16991 [math.NA] (2024).