Integral dielectric kernel approach to modelling RF heating in toroidal plasmas

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Outline

- Background, reminder of (mixed FEM-) spectral approaches
- Motivation for configuration space approach
- Integral kernels in configuration space:
  - Homogeneous plasmas
  - Tokamaks: mixed “toroidal spectral - poloidal configuration space” formulation
  - Tokamaks and stellarators: full configuration space formulation
- Prospects & conclusions

Background

- Kinetic description of hot plasma HF response in realistic toroidal geometries is challenging: rotational transform, curved equilibrium $\boldsymbol{B_0},\ \nabla_{//} B_0 \neq 0$.
- Very different world from $\perp$ - stratified, straight-$\boldsymbol{B_0}$ equilibria

  $\Rightarrow$ Traditional approach to realistic full-wave modelling of wave heating in tokamaks and stellarators relies on Fourier expansions of the HF fields along 2 or 3 spatial coordinates.

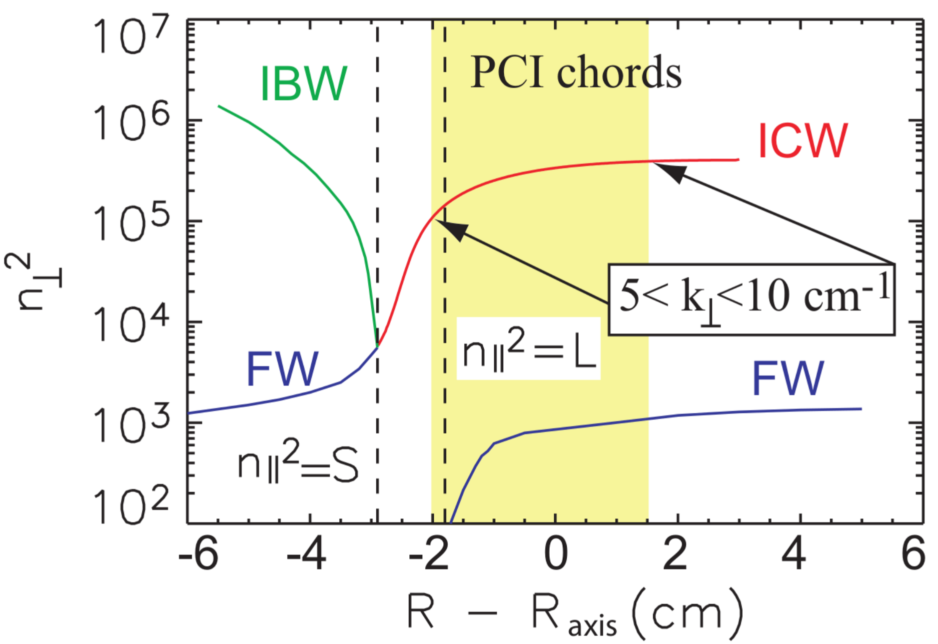
- Indeed allows convenient kinetic theoretical treatment of wave dispersion along curved $\boldsymbol{B_0}$;

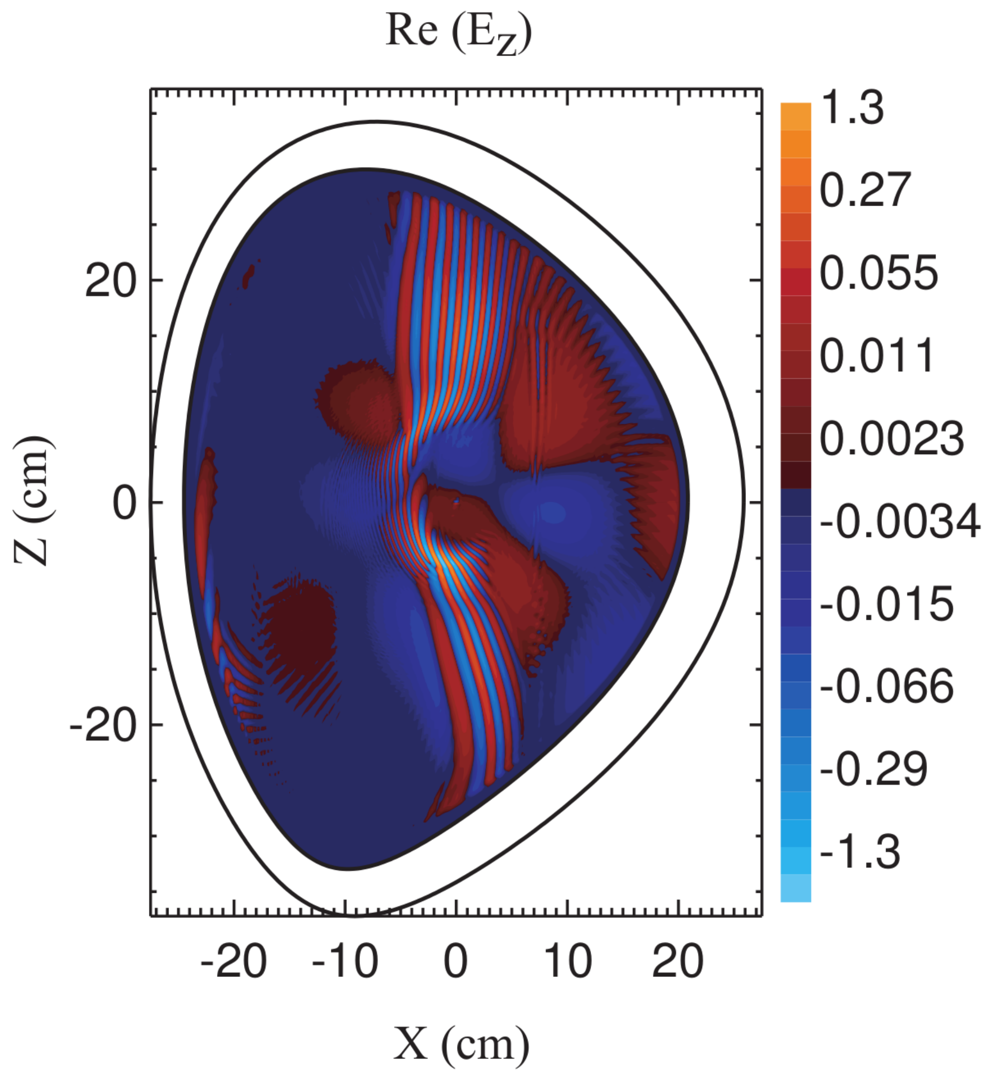
  $\Rightarrow$

  - Mixed spectral-finite element or fully spectral numerical formulations
  - Plasma described by dielectric tensor formulated in Fourier space involving the well-known $Z^{\{\alpha\}}$ dispersion functions for Maxwellians
  - Used in most codes: TORIC, CYRANO, EVE, AORSA, ...

Background: sample advanced ICRF simulations - TORIC

- ICRF, ICW mode conversion in Alcator C-Mod (IBW also excited)
- 240 radial FE, 255 poloidal & 1 toroidal Fourier modes

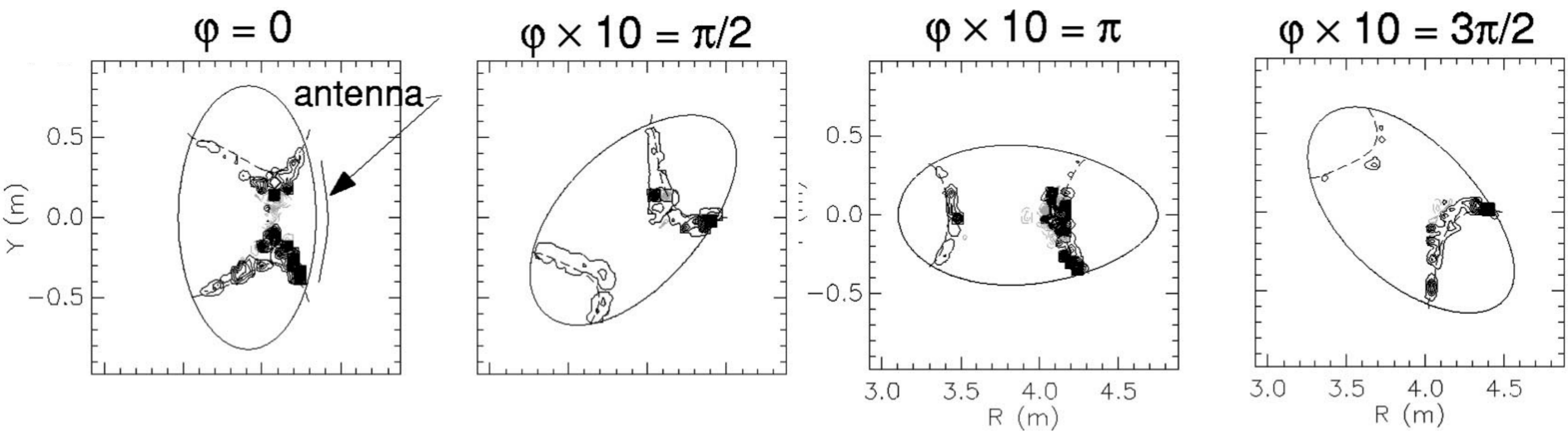




from [Bonoli *et al* 2007]

Background: sample advanced ICRF simulations - AORSA

- ICRF minority ion heating in LHD, 5% H in He
- Fully spectral in cylindrical coordinates, all-orders FLR i.e. arbitrary $k_{\perp}\rho_{LT}$
- 50 modes in R and Y, 16 strongly coupled toroidal modes
- 10 simulations over 1 helical field period allow reconstruction over whole device



Minority heating contours at 4 toroidal positions

from [Jaeger *et al* 2002]

Reference set of equations: linearized Vlasov - Maxwell system

- Maxwell-Vlasov system (frequency domain), ‘weak’ form:

$$\frac{\mathrm{i}}{2}\int_{\mathcal{V}}\left[\frac{1}{\omega\,\mu_0}(\nabla\times\boldsymbol{F})^*.(\nabla\times\boldsymbol{E})-\omega\,\varepsilon_0\,\boldsymbol{F}^*.\boldsymbol{E}\right]\,\mathrm{d}r^3+\sum_\beta\mathcal{W}_{\boldsymbol{FE}\beta}=-\frac{1}{2}\int_{\mathcal{V}}\boldsymbol{F}^*.\boldsymbol{j}_S\,\mathrm{d}r^3$$

(**E**: RF electric field, **F**: arbitrary test function field, **F** ≡ **E** ⇒ Poynting’s theorem)

- This formulation emphasizes the dielectric response of each species β:

$$\boxed{\mathcal{W}_{\boldsymbol{FE}\beta}}=\frac{1}{2}\int_{\mathcal{V}}\boldsymbol{F}^*\cdot\boldsymbol{j}_\beta\,\mathrm{d}r^3=\frac{q_\beta}{2}\int_{\mathcal{V}}\mathrm{d}r^3\int\mathrm{d}v^3\,f_\beta\boldsymbol{F}^*\cdot\boldsymbol{v}.$$

... rather than the RF current density

$$\boldsymbol{j}_\beta(\boldsymbol{r})\equiv\boldsymbol{\sigma}_\beta\cdot\boldsymbol{E}=q_\beta\int f_\beta\boldsymbol{v}\,\mathrm{d}v^3$$

- Vlasov HF perturbed distribution function:

$$f_\beta(\boldsymbol{r},\boldsymbol{v})=-\frac{q_\beta}{m_\beta}\int_{-\infty}^{t}\mathrm{e}^{-\mathrm{i}\omega(t'-t)}[\boldsymbol{E}(\boldsymbol{r}')+\boldsymbol{v}'\times\boldsymbol{B}(\boldsymbol{r}')]\cdot\frac{\partial f_{0\beta}}{\partial\boldsymbol{v}'}\,\mathrm{d}t'$$

- Interests:

  Theory: facilitates consistent treatment of geometry
  Applications: ideally suited for implementation in FEM codes
  & extraction of power balance

Reminder: 2D-spectral tokamak theory - Notations

- Field components: left (+) and right (-) circular polarizations, parallel (//)

$$\boldsymbol{E} = E_+ \,\boldsymbol{e}_+ + E_- \,\boldsymbol{e}_- + E_{//} \,\boldsymbol{e}_{//} = \sum_{L=-1}^{+1} E_{\mathcal{L}} \,\boldsymbol{e}_{\mathcal{L}} \,, \qquad \boldsymbol{F} = \sum_{L'=-1}^{+1} F_{\mathcal{L}'} \,\boldsymbol{e}_{\mathcal{L}'}$$

$\mathcal{L}$	$+$	$-$	$//$
L	$+1$	-1	0

- '$v_{//}^{\alpha}$ index': $\alpha = \delta_{L,0} + \delta_{L',0}$

  $\alpha = 0$: cyclotron and TTMP

  $\alpha = 2$: Landau

  $\alpha = 1$: mixed Landau-TTMP

- Lowest order FLR: only 'diagonal' contributions $L = L'$, $\alpha = 2\delta_{L,0}$

  i.e. the bilinear dielectric response only involves $F_{\mathcal{L}} \, E_{\mathcal{L}}$ terms

- We use p for the cyclotron harmonic index:

  $p = +1$: ion fundamental, $p = 0$: Landau, $p = -1$: electron fundamental

Reminder: 2D-spectral tokamak theory (TORIC, CYRANO, EVE,... codes)

- Mode expansions:

$$E_{\mathcal{L}} = \sum_{m,n} E_{m_1,n}(\rho)\, e^{i(m_1\theta + n\varphi)}, \qquad F_{\mathcal{L}'} = \sum_{m_2,n} F_{m_2,n}(\rho)\, e^{i(m_2\theta + n\varphi)}$$

- The plasma response exhibits strong coupling between poloidal modes: to lowest order FLR,

$$\mathcal{W}^{(n)}_{\boldsymbol{FE}\beta} = \sum_{p=-1}^{+1} \sum_{m_1,\, m_2=-\infty}^{+\infty} \mathcal{W}^{p}_{21\beta}$$

Matrix elements: Fourier transforms of Maxwellian dispersion functions $Z^{\{\alpha\}}$

$$\mathcal{W}^{p}_{21\beta} = -\mathrm{i}\pi\varepsilon_0 \sum_{L=-1}^{+1} \delta_{L,p} \int \mathrm{d}\rho\, F^{*}_{\mathcal{L},m_2} \left\{ \int \mathcal{J}\, \frac{2^{\alpha/2}\,\omega_{\mathrm{p}}^2}{\left|k_{//\bar{m},n}\right| v_{\mathrm{T}}}\, Z^{\{\alpha\}}\!\left(\frac{\omega - p\omega_{\mathrm{c}}}{\left|k_{//\bar{m},n}\right| v_T} \right) \mathrm{e}^{\mathrm{i}(m_1-m_2)\theta}\, \mathrm{d}\theta \right\} E_{\mathcal{L},m_1}$$

$$(\alpha = 2\delta_{L,0})$$

- The // wavenumber varies with position:

- with standard angle variables, $k_{//}(\rho,\theta) = \bar{m}\sin\Theta/N_\theta + n\cos\Theta/R$

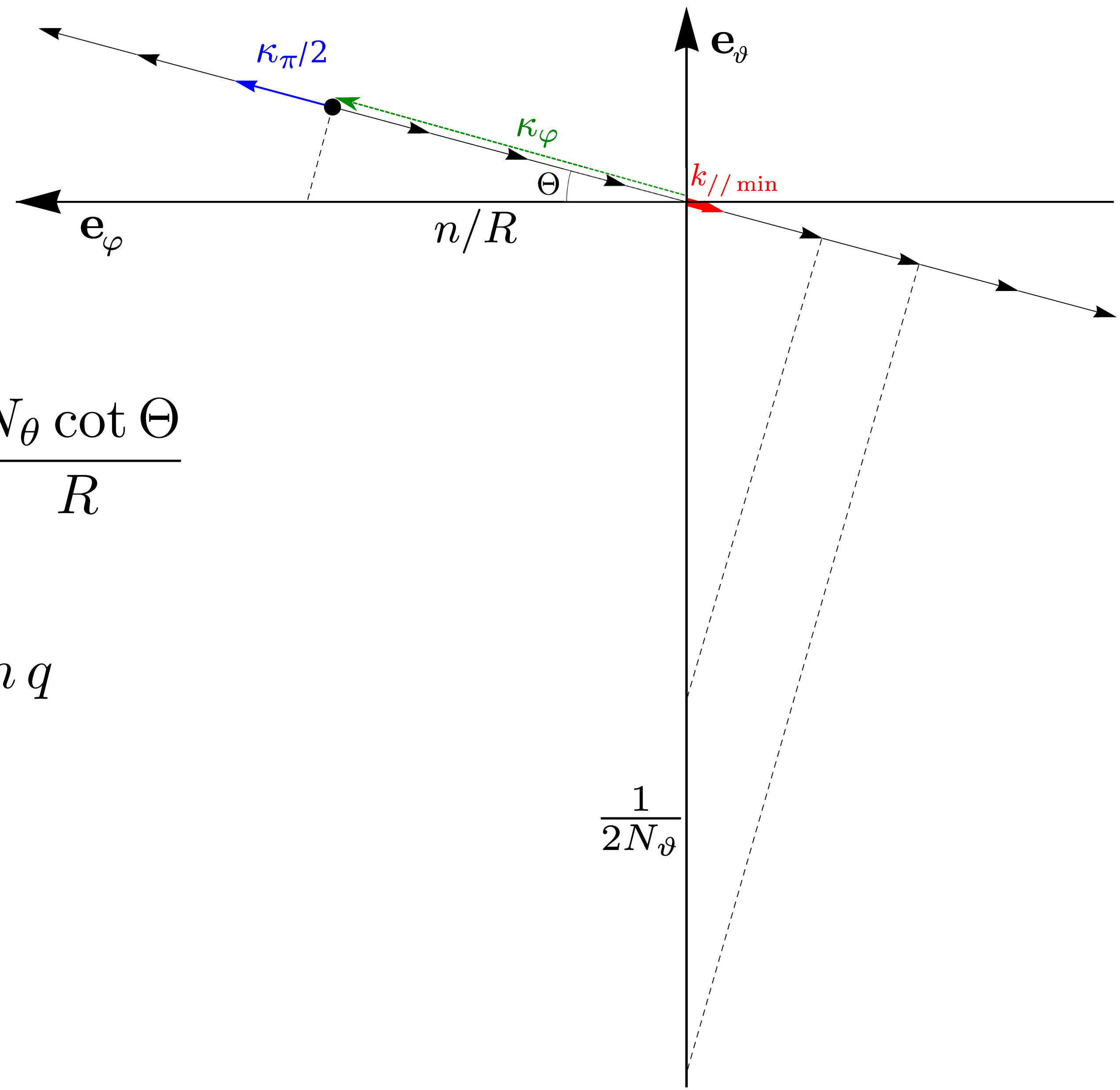
- with the straight-magnetic-field angles $(\bar{\theta},\bar{\varphi})$ of [Lamalle 2006],

$$k_{//}(\rho) = (\bar{m} + nq)/H(\rho)$$

- Hamiltonian and fully consistent gc theories lead to the symmetric index $\boxed{\bar{m} = (m_1+m_2)/2}$, which guarantees a purely resonant wave absorption

[Lamalle 1997, Dumont 2009], CYRANO and EVE codes

Features of the $k_{//}$ set in tokamak geometry



- Write

$$k_{//} = \frac{\kappa_\pi}{2}\,(m_1 + m_2 + \kappa)$$

- Standard angles (θ, φ):

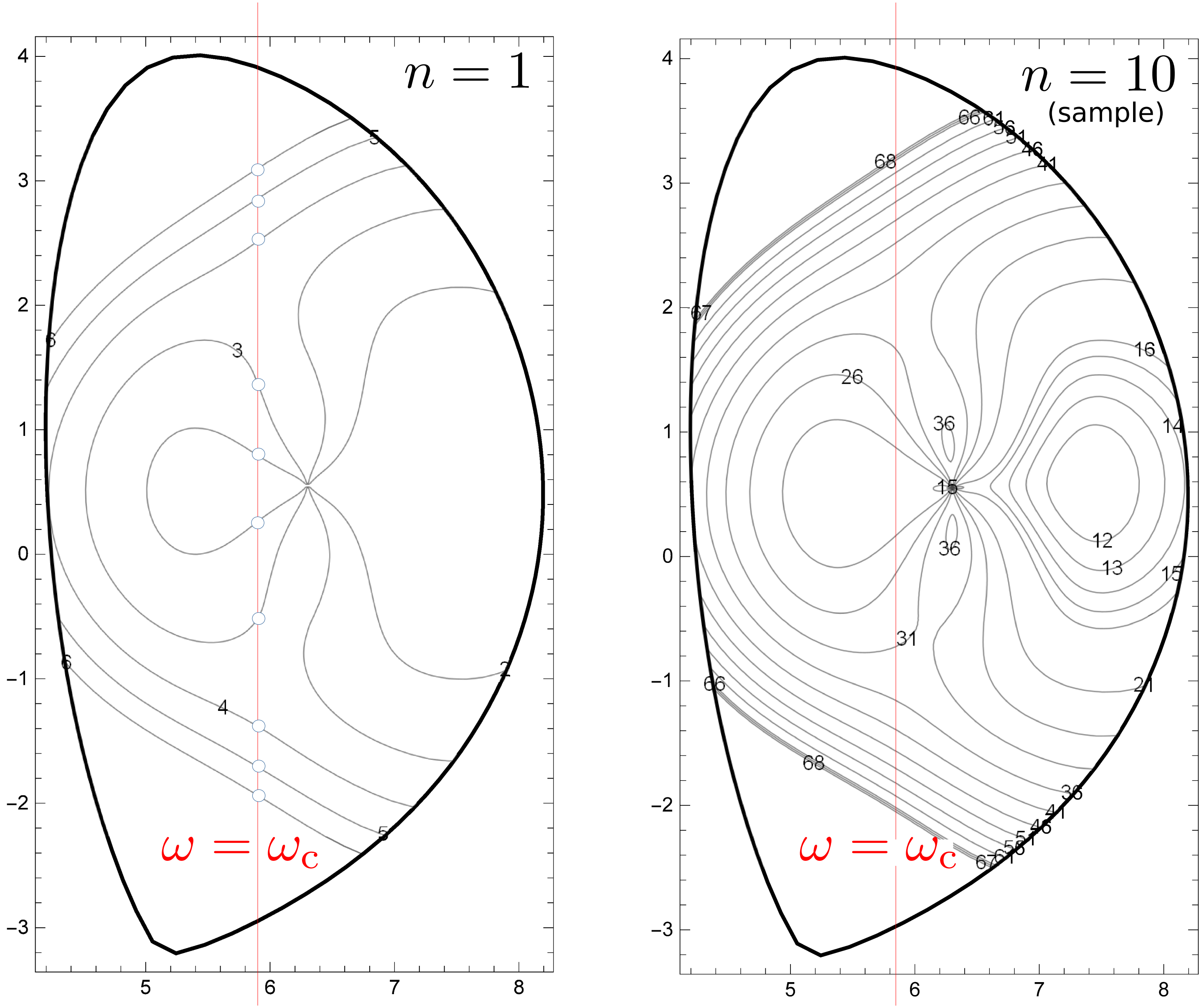
$$\kappa_\pi = \frac{\sin\Theta}{N_\theta}, \qquad \kappa = 2n\frac{N_\theta \cot\Theta}{R}$$

- 'Constant-$k_{//}$' angles $(\bar{\theta}, \bar{\varphi})$:

$$\kappa_\pi = 1/H(\rho), \qquad \kappa = 2n\,q$$

- κ turns out a key parameter
- Whenever $\boxed{\kappa \in \mathbb{Z}}$ the poloidal spectrum includes a $k_{//} = 0$

Features of the $k_{//}$ set in tokamak geometry: standard θ, φ

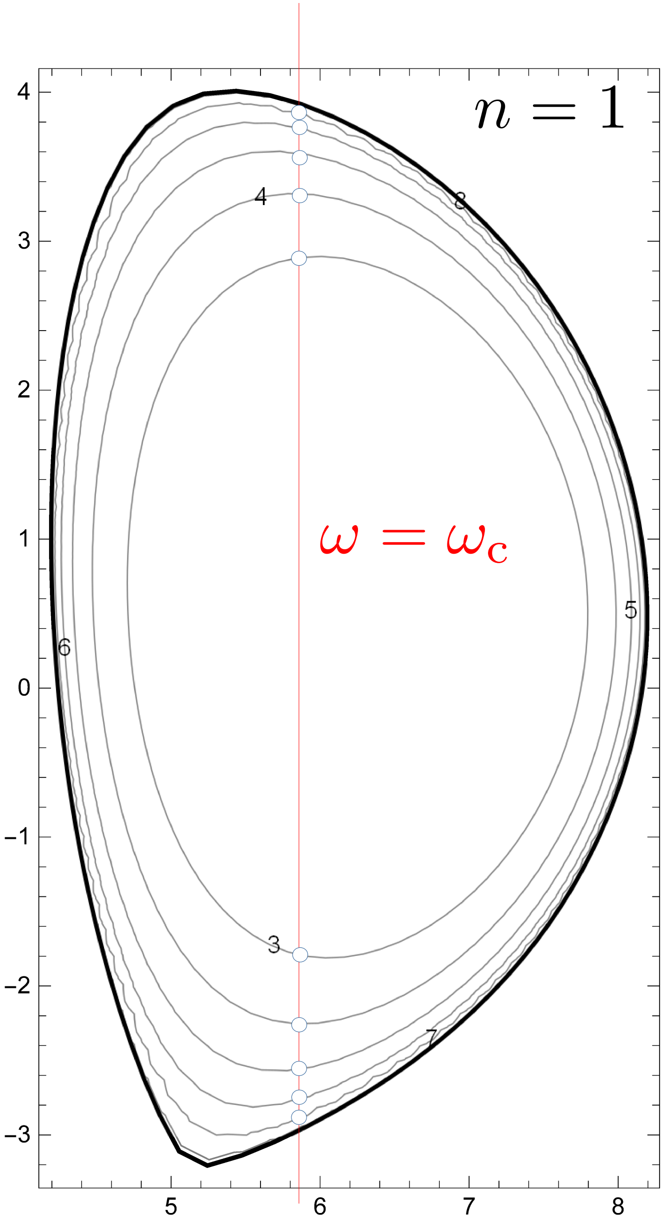


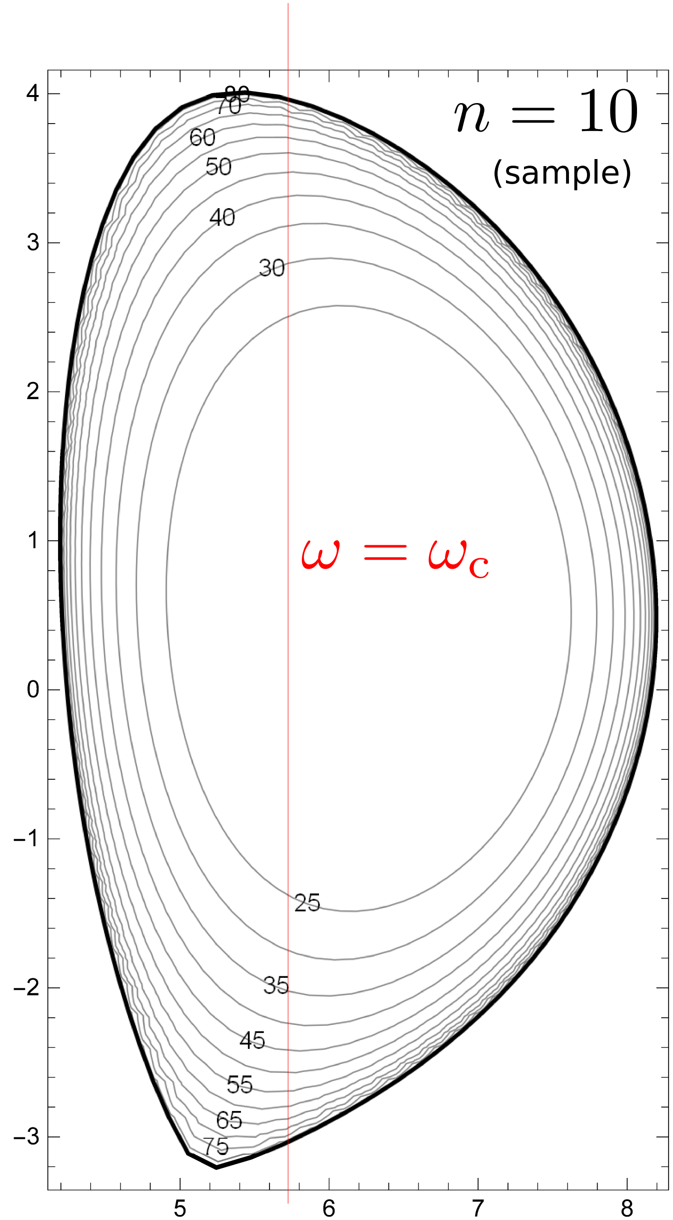
- Despite $T_i \neq 0$, a singular cold resonance $k_{//}=0$ occurs at each intersection ◯
- More and more numerous as *n* ↑
- Embedded in all 2D-spectral codes
- Likely issue: uncontrolled FFTs of $1/[\omega_c(\theta) - \omega]$
- Axisym *n*=0: cold resonance overall for $\bar{m} = 0$!

The surfaces of integer $\kappa = 2n \dfrac{N_\theta \cot \Theta}{R}$

ITER equilibrium (2.65T, 7.5MA)

Quite a different picture with 'constant-$k_{//}$' angles $\bar{\theta}$, $\bar{\varphi}$





- Singular cold resonance $k_{//}$=0 at each intersection with a set of rational surfaces
- Same issue as std angles but easier to handle

Surfaces of integer $\kappa = 2n\,q$

ITER equilibrium (2.65T, 7.5MA)

Now, to the point: Integral kernels in configuration space

- Ongoing research: alternative approach,

  dielectric response as a nonlocal integral operator in physical space,

  involving new Maxwellian '*kernel dispersion functions*'.

- Max analytical developments
  $\Rightarrow$ extra physics insight and faster numerical simulations.

- Current emphasis on long-range dispersion effects along $\boldsymbol{B_0}$, outstanding in applications. First implementation in progress [Reman *et al*, Tuesday].

- N.B. For the sake of clarity, presentation only shows lowest order FLR.

Advantages of the configuration space integral approach

- Leaves complete freedom of choice for numerical discretization.

- $\Rightarrow$ Enables FEM methods in 2D and 3D to model wave propagation and absorption in hot inhomogeneous fusion plasmas;

- Enables local mesh refinements (ruled out with spectral methods), essential to address FLR effects in 2D / 3D;

- Better suited field representations to deal with FLR in toroidal geometry;

- Straightforward connection with RF antenna models based on the FEM.

$\Rightarrow$ Main goals of our ongoing project:

- Efficient implementation in new full-wave code & existing FEM packages;

- Validation, demonstration of attractiveness, model RF heating in tokamaks and stellarators.

Earlier work on the configuration space approach

- Sauter & Vaclavik 1992, 1994, Smithe *et al* 1997: $\perp$-stratified plasma, focus on $\perp$ nonlocal effects, spectral in // direction

- Meneghini, Shiraiwa & Parker 2009: LHCD, integral treatment of Landau damping, iterative solution

- Svidzinski 2016: very general approach, hot conductivity kernel evaluated numerically by orbit integration

- Fukuyama, 2019 RFPPC: see next presentation

- Lamalle, 2019 RFPPC, 2023 EFTC, 2024 Varenna: tokamak theory, // treatment

- Machielsen, Rubin & Graves 2023: full FLR theory for homogenous plasmas, both // and $\perp$ treatments. Applied to $\perp$ but not yet to //.

Other recent treatments of // dispersion by iterative methods:
Vallejos *et al* 2018, Zaar *et al* 2024

- In common: Maxwell-Vlasov system, weak form:

$$\frac{\mathrm{i}}{2}\int_{\mathcal{V}}\left[\frac{1}{\omega\,\mu_0}(\nabla\times\boldsymbol{F})^*.(\nabla\times\boldsymbol{E})-\omega\,\varepsilon_0\,\boldsymbol{F}^*.\boldsymbol{E}\right]\mathrm{d}r^3+\sum_{\beta}\mathcal{W}_{\boldsymbol{FE}\beta}=-\frac{1}{2}\int_{\mathcal{V}}\boldsymbol{F}^*.\,\boldsymbol{j}_S\,\mathrm{d}r^3$$

- Plasma response, 0[th] order FLR, showing homogeneous plasma case,

  - Usual spectral formulation: local in $k_{//}$ space

$$\mathcal{W}_{\boldsymbol{FE}\beta}=-\mathrm{i}\pi\varepsilon_0\sum_{L=-1}^{1}\delta_{L,p}\,2^{\alpha/2}\int\mathrm{d}r_\perp^2$$
$$\int_{-\infty}^{+\infty}\mathrm{d}k_{//}\;\tilde{F}^*_{\mathcal{L}}(k_{//})\,\frac{\omega_{\mathrm{p}}^2}{k_{//}v_{\mathrm{T}}}\boxed{Z^{\{\alpha\}}\!\left(\frac{\omega-p\omega_{\mathrm{c}}}{k_{//}v_{\mathrm{T}}}\right)}\,\tilde{E}_{\mathcal{L}}(k_{//})\qquad(\alpha=2\delta_{L,0})$$

  - Configuration space formulation: obtained by back Fourier transform of $\tilde{E}_{\mathcal{L}}(k_{//})$

$$\mathcal{W}_{\boldsymbol{FE}\beta}=-\frac{\mathrm{i}\varepsilon_0}{2}\sum_{L=-1}^{1}\delta_{L,p}\,2^{\alpha/2}\int\mathrm{d}r_\perp^2$$
$$\int\mathrm{d}z\int\mathrm{d}z'\;F^*_{\mathcal{L}}(z')\,\frac{\omega_{\mathrm{p}}^2}{v_{\mathrm{T}}}\boxed{\Upsilon_\alpha\!\left(\frac{\omega-p\,\omega_{\mathrm{c}}}{2\,v_{\mathrm{T}}}\,|z'-z|\right)}\,E_{\mathcal{L}}(z)\qquad(\alpha=2\delta_{L,0})$$

Configuration space approach: infinite //-homogeneous plasmas

- Plasma response, showing 0th order FLR: involves nonlocal integrals (z, z') along magnetic field lines

$$\mathcal{W}_{\boldsymbol{FE}\beta} = -\frac{\mathrm{i}\varepsilon_0}{2} \sum_{L=-1}^{1} \delta_{L,p}\, 2^{\alpha/2} \int \mathrm{d}r_\perp^2\, \frac{\omega_\mathrm{p}^2}{v_\mathrm{T}} \int \mathrm{d}z \int \mathrm{d}z'\, F_\mathcal{L}^*(z')\, \boxed{\Upsilon_\alpha(\lambda\,|z'-z|)}\, E_\mathcal{L}(z) \quad (\alpha = 2\delta_{L,0})$$

$$\lambda = \frac{\omega - p\,\omega_\mathrm{c}}{2\,v_\mathrm{T}}$$

The kernel dispersion functions (KDF) are derived from the usual PDF:

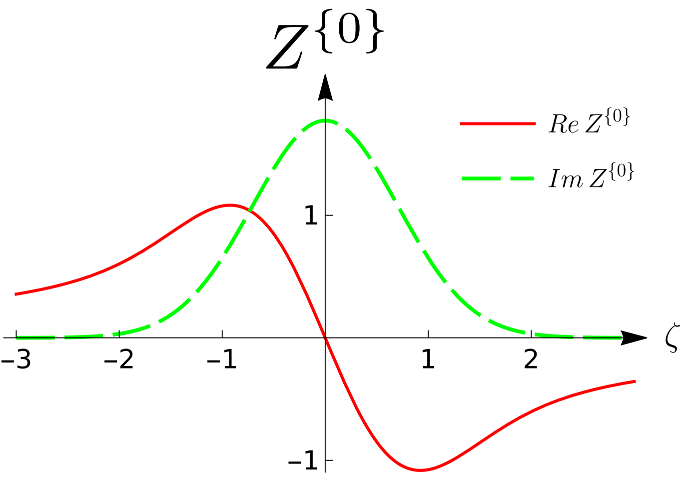
$$\boxed{\Upsilon_\alpha(\xi) = \frac{1}{\pi}\int_0^{+\infty} Z^{\{\alpha\}}\left(\frac{2\,\xi}{t}\right) \left(\begin{array}{c} \cos t \\ \mathrm{i}\,\sin t \end{array}\right) \frac{\mathrm{d}t}{t}, \qquad \alpha \left\{\begin{array}{c} \text{even} \\ \text{odd} \end{array}\right\}, \qquad \mathrm{Im}\,\xi > 0}$$

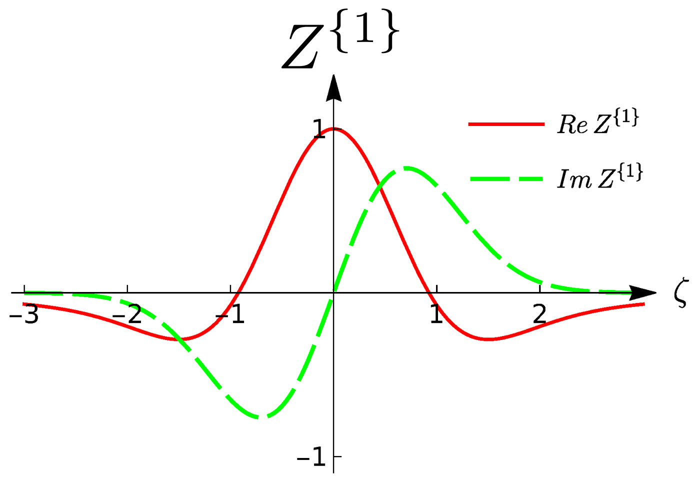
- $Z^{\{\alpha\}}$: standard plasma dispersion functions
- Odd α kept for completeness, $\alpha = 1$ enters the FLR theory
- The form of Υ_0 appears in Svidzinski (2016)'s conductivity kernel.

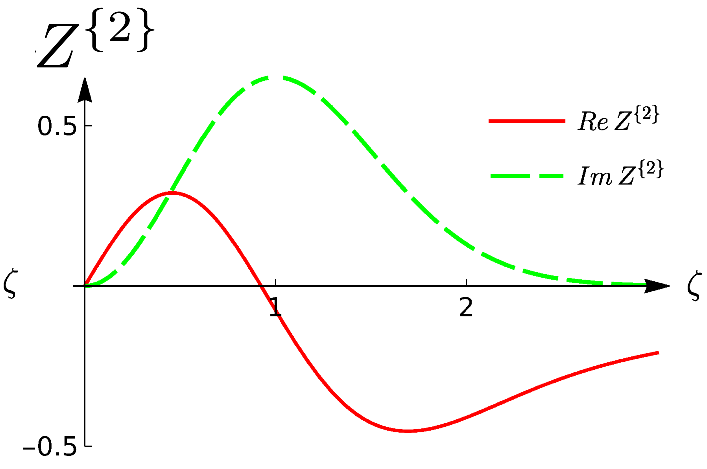
Lamalle (2023, 2024), equivalent to Machielsen *et al* (2023) S_1, S_2, S_3 but ≠ def.

Correspondence

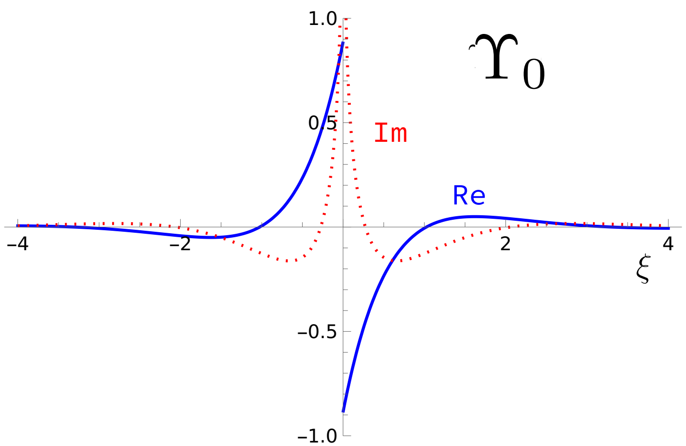
- Plasma dispersion functions: at $k_{//} = 0$, $\mathrm{Im}\, Z^{\{\alpha\}}(\pm\infty) = 0$



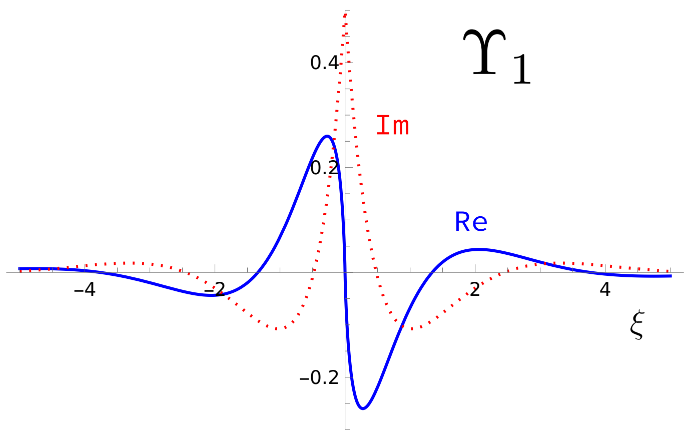




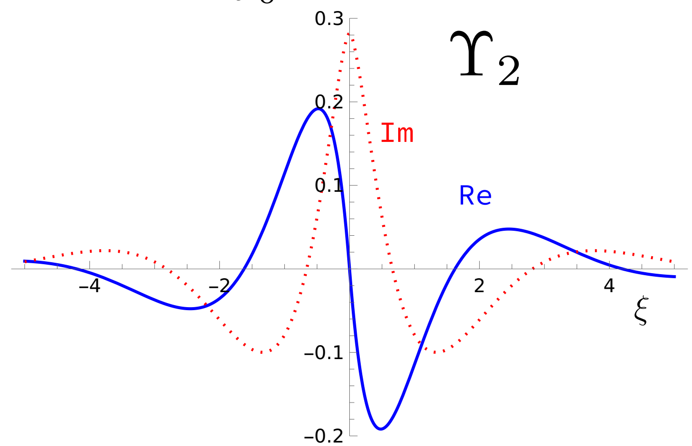
- Kernel dispersion functions: on uniform RF field, $\int_0^\infty \mathrm{Im}\,\Upsilon_\alpha(\xi)\,\mathrm{d}\,\xi = 0$



Cyclotron (p≠0), TTMP
Im log singular at $\xi = 0$



Mixed TTMP-Landau



Landau (p=0)

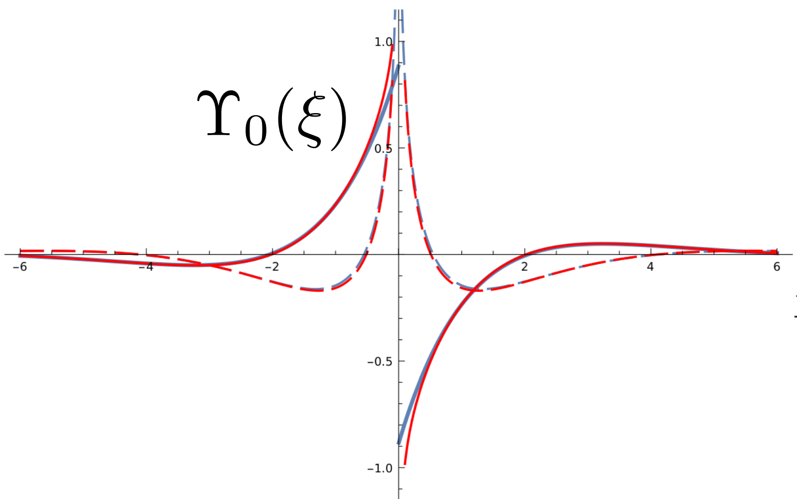
Detailed analytical study of the KDF

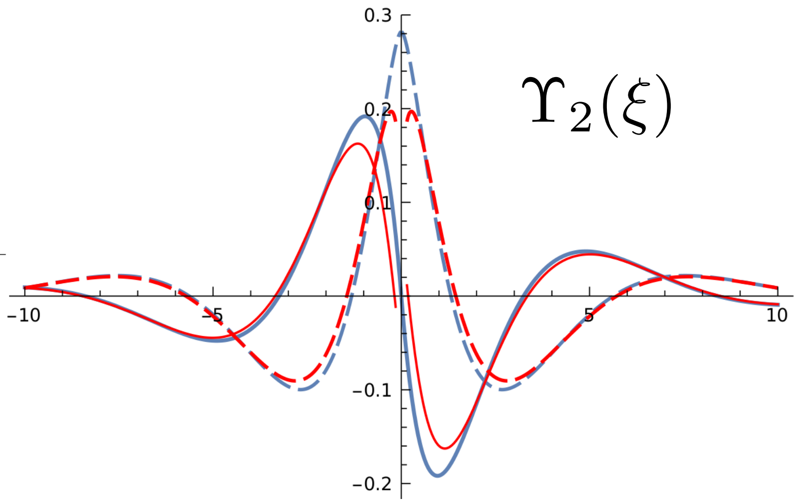
- Definition ensures genuine functions of a single variable
- Many analytical properties established and exploited in the applications:
  Series expansions, integral representation, recurrences, differential equation, ...

$$\Upsilon_\alpha(\xi) = \frac{\mathrm{i}}{2\sqrt{\pi}} \int_{\to 0}^{+\infty} \mathrm{e}^{-u+2\mathrm{i}\xi/\sqrt{u}}\, u^{\alpha/2-1}\, \mathrm{d}u\,, \quad \mathrm{Im}\,\xi > 0$$

- Leading asymptotic expansion emphasizes their compact support:

$$|\Upsilon_\alpha(\xi)| \sim \exp\{-3\,|\xi|^{2/3}/2\}\, |\xi|^{(\alpha-1)/3}/\sqrt{3}, \quad \xi \to \pm\infty$$





Blue: series,
Red: asymptotic exp.

- Latest result: the double // integral over pairs of FE basis functions are now evaluated semi-analytically ⇒ code simplification and strong acceleration ahead!

‘Lumping’ the FEM system matrix

$$\Upsilon_\alpha\left(\frac{\omega - p\,\omega_\mathrm{c}}{v_\mathrm{T}}\,\frac{|z'-z|}{2}\right)$$

- // - homogeneous plasma: significant // nonlocal cyclotron interaction $(\alpha = 0)$ over

$$|z'-z| \lesssim \frac{K\,v_\mathrm{T}}{\omega - p\,\omega_\mathrm{c}}, \qquad K \sim \text{"}8\text{"}$$

Careful treatment at resonance layer...

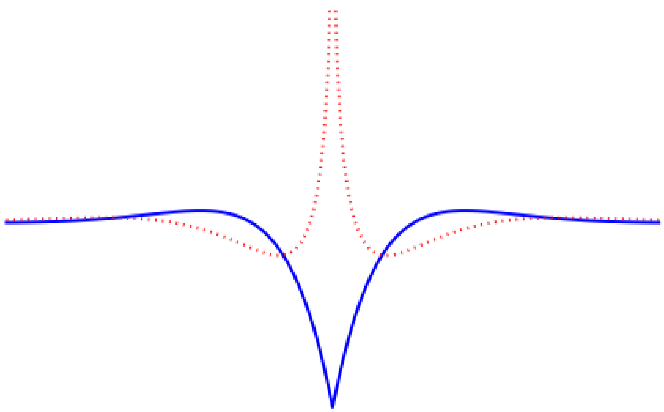
- Nonlocal Landau interaction $(\alpha = 2)$ over

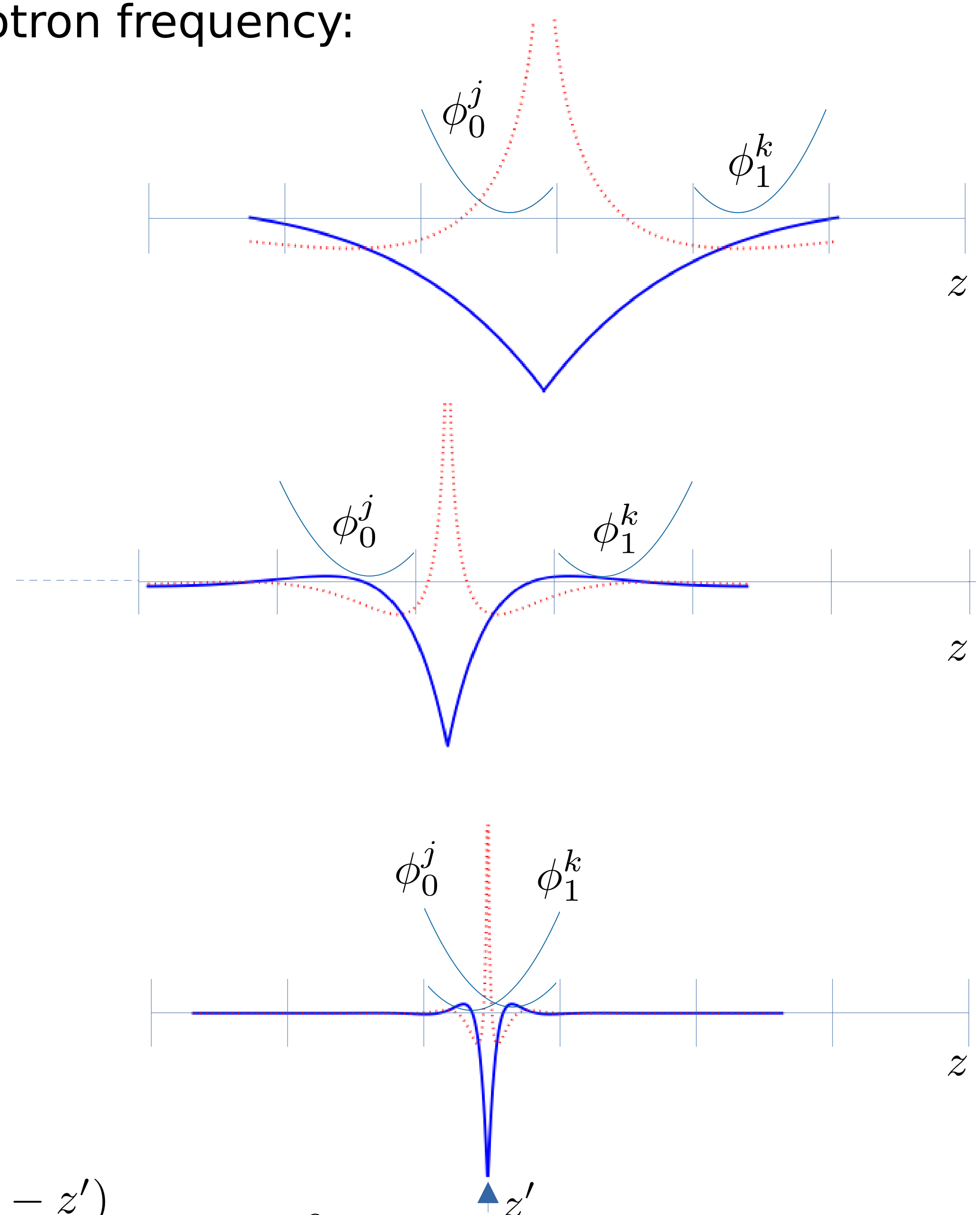
$$|z'-z| \lesssim K\,\frac{v_\mathrm{T}}{\omega}, \qquad K \sim \text{"}8\text{"}$$

Cyclotron interaction - FEM numerical treatment

Decreasing values of T_i at fixed z' and cyclotron frequency:

Υ_0 vs $|z - z'|$:





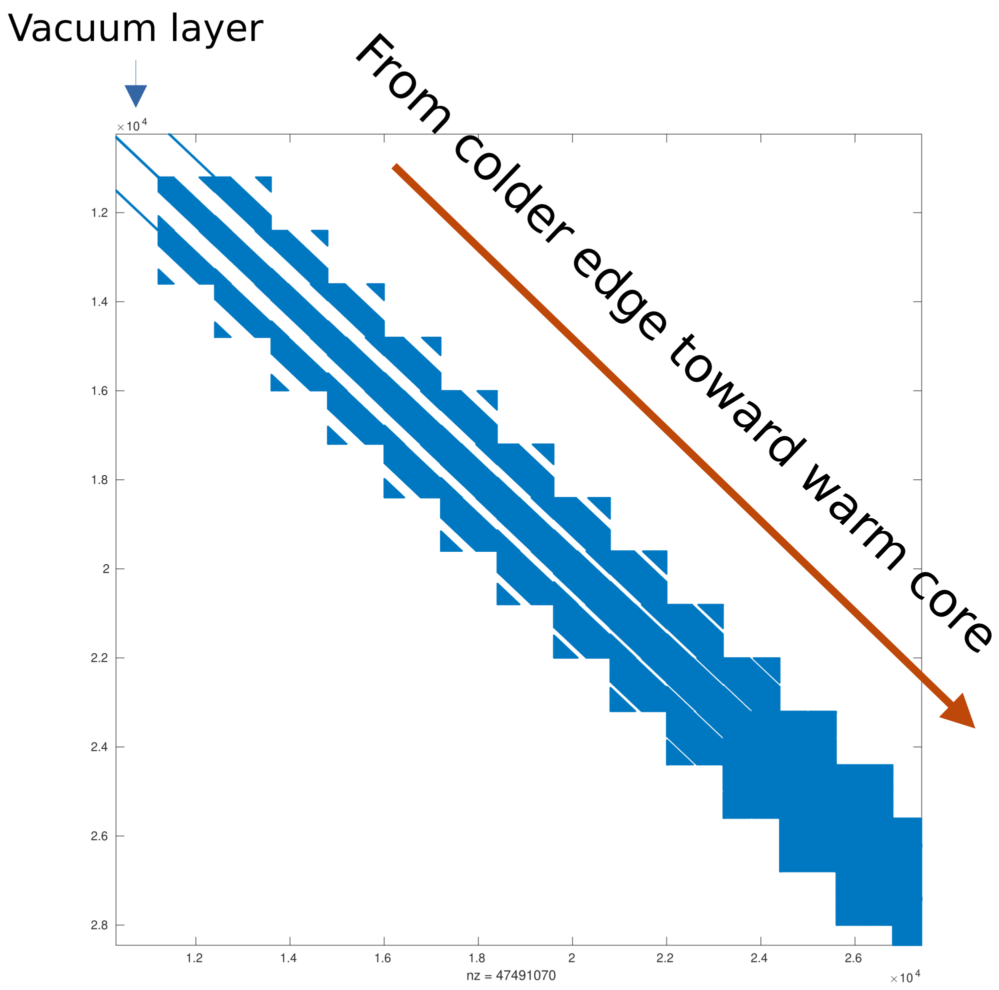
Cold limit: $v_{\mathrm{T}}^{-1}\,\Upsilon_0\left(\dfrac{\omega - p\,\omega_{\mathrm{c}}}{2\,v_{\mathrm{T}}}\,|z - z'|\right) \sim -\dfrac{\delta(z - z')}{\omega - p\omega_{\mathrm{c}}}, \quad v_{\mathrm{T}} \to 0$

FEM global matrix: sparsity pattern on 2D slab code

Minority H in D

Element numbering sequential along field lines, then across

Domain periodic in // direction



[Reman *et al, Tuesday*]

Tokamak scheme of derivation: simply ‘back-Fourierize’

Tokamak spectral theory

$$E_{\mathcal{L}} = \sum_{m,n} E_{m,n}(\rho)\, e^{i(m\theta+n\varphi)}$$

$$Z^{\{\alpha\}}\left(\frac{\omega - p\omega_c}{\left|k_{//\bar{m},n}\right| v_T}\right)$$

- More easily said than done…
- $\Rightarrow$ This integral theory is mathematically equivalent with the spectral one, common physics content.

Remove pol expansion: $E_{m,n}(\rho) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{+\pi} E_n(\rho,\theta)\,\mathrm{e}^{-\mathrm{i}m\theta'}\,\mathrm{d}\theta'$

Mixed tor spectral - pol configuration space

$$E_{\mathcal{L}} = \sum_{n} E_n(\rho,\theta)\, e^{in\varphi}$$

PKDF $\quad \Xi_\alpha(\chi, 2nq, \xi_p)$

$$\xi_p = \frac{\omega - p\,\omega_c}{\left|\kappa_\pi\right| v_T}$$

Remove tor expansion:

$$E_n(\rho,\theta) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} E(\rho,\theta,\varphi)\,\mathrm{e}^{-\mathrm{i}n\varphi'}\,\mathrm{d}\varphi'$$

Full configuration space

$$E_{\mathcal{L}} = E_{\mathcal{L}}(\rho,\theta,\varphi)$$

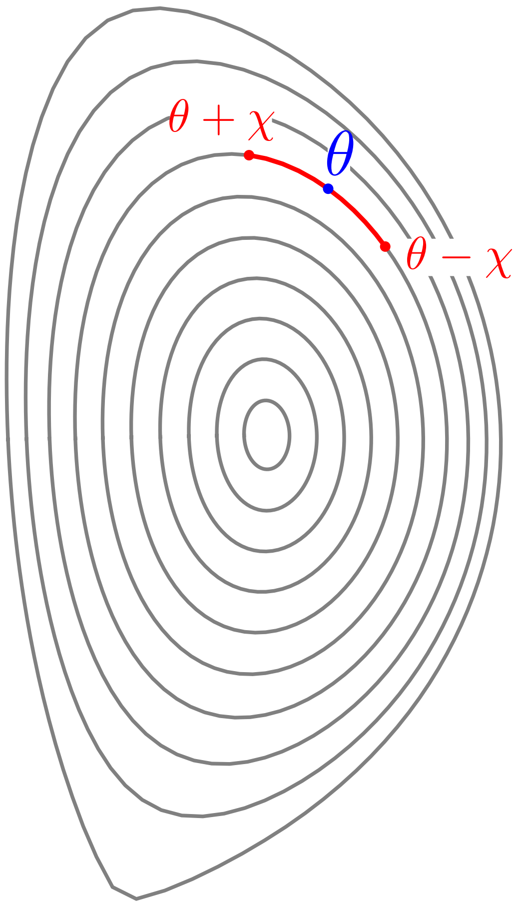
‘Field-line’ KDF

$$\Upsilon_\alpha\left(\frac{\omega - p\,\omega_c}{v_T}\,\frac{\Delta s}{2}\right)$$

Tokamak: mixed toroidal spectral - poloidal configuration space

- Applies to single toroidal Fourier modes n, 2D FE simulations on meridian section
- Plasma response (showing 0th order FLR) involves a nonlocal poloidal integral (χ):

$$\mathcal{W}^{(n)}_{\boldsymbol{FE}\beta} = 2\pi \int \mathrm{d}\rho \sum_{p=-1}^{+1} \sum_{L=-1}^{+1} \int_{-\pi}^{\pi} \mathcal{J} \int_{-\pi}^{\pi} F^*_{\mathcal{L}}(\theta+\chi) \boxed{\mathcal{K}^p_{\mathcal{L}\mathcal{L}}(\theta,\chi)} E_{\mathcal{L}}(\theta-\chi) \, \mathrm{d}\chi \, \mathrm{d}\theta$$



The equilibrium parameters are evaluated at the mid-poloidal θ between field and test points

Tokamak: mixed toroidal spectral - poloidal configuration space

- Plasma response (showing 0[th] order FLR) involves a nonlocal poloidal integral (χ):

$$\mathcal{W}^{(n)}_{\boldsymbol{F}\boldsymbol{E}\beta} = 2\pi \int \mathrm{d}\rho \sum_{p=-1}^{+1} \sum_{L=-1}^{+1} \int_{-\pi}^{\pi} \mathcal{J} \int_{-\pi}^{\pi} F^{*}_{\mathcal{L}}(\theta+\chi) \boxed{\mathcal{K}^{p}_{\mathcal{L}\mathcal{L}}(\theta,\chi)} E_{\mathcal{L}}(\theta-\chi)\,\mathrm{d}\chi\,\mathrm{d}\theta$$

$$\boxed{\mathcal{K}^{p}_{\mathcal{L}\mathcal{L}}(\theta,\chi)} = -\frac{\mathrm{i}\varepsilon_0}{2}\,\delta_{L,p}\,2^{\alpha/2}\,(\mathrm{sign}\,\kappa_\pi)^{\alpha}\,\frac{\omega_{\mathrm{p}}^2}{|\kappa_\pi|\,v_{\mathrm{T}}}\,\boxed{\Xi_\alpha(\chi,\kappa,\xi_p)} \quad (\alpha = 2\delta_{L,0}).$$

$$\xi_p = \frac{\omega - p\,\omega_{\mathrm{c}}}{|\kappa_\pi|\,v_{\mathrm{T}}}$$

- Based on poloidal kernel dispersion functions (PKDF) of 3 independent variables:

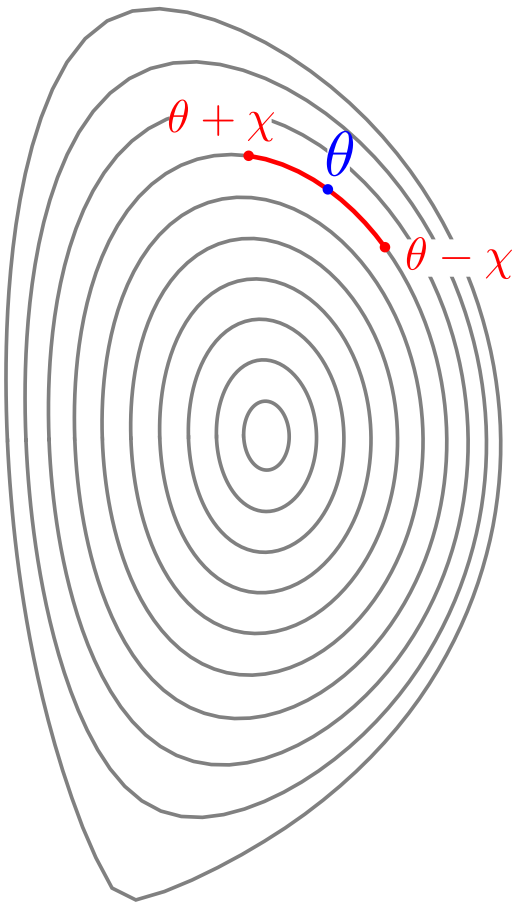
$$\boxed{\Xi_\alpha(\chi,\kappa,\xi) = \frac{1}{\pi}\sum_{M=-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}M\chi}\,\frac{[\mathrm{sign}\,(M+\kappa)]^{\alpha}}{|M+\kappa|}\,Z^{\{\alpha\}}\!\left(\frac{2\,\xi}{|M+\kappa|}\right), \quad \mathrm{Im}\,\xi > 0}$$

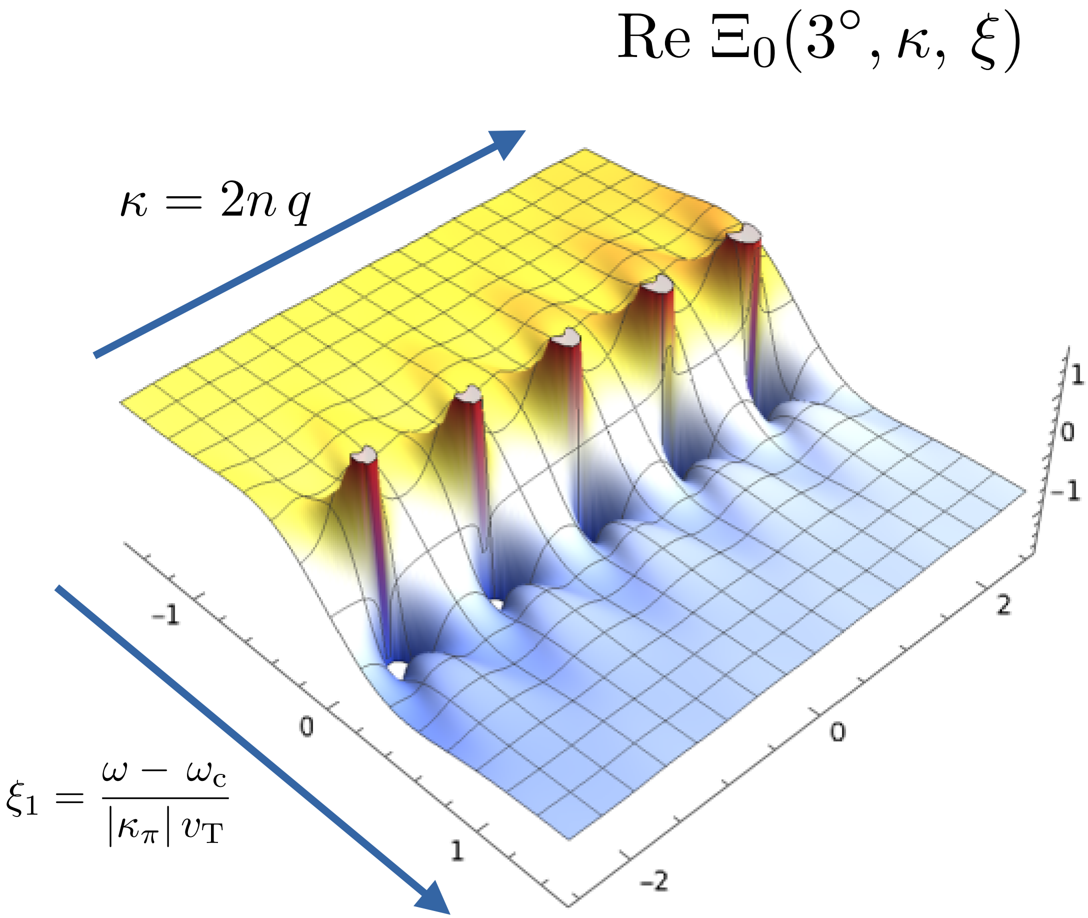
- Equilibrium parameters are evaluated at mid-point θ between field and test points

- Std angles: κ_π local poloidal curvature, $\kappa = 2nN_\theta \cot\Theta/R$
- CKP angles: $\kappa_\pi = 1/H$, $\kappa = 2nq$

Tokamak: mixed toroidal spectral - poloidal configuration space

- Applies to single toroidal Fourier modes n, 2D FE simulations on meridian section
- Sample plot of the cyclotron PKDF:





- Math properties well documented in view of applications
- Handling singularities at $\kappa \in \mathbb{Z}$: causal treatment of θ integral

Tokamak: full configuration space result

- The toroidal Fourier expansion is removed and the previous expressions simplified.
- Plasma response, showing 0th order FLR: involves nonlocal integrals (*s, s'*) along magnetic field lines,

$$\mathcal{W}_{\boldsymbol{FE}\beta} = -\frac{\mathrm{i}\varepsilon_0}{2} \sum_{L=-1}^{1} \delta_{L,p}\, 2^{\alpha/2} \int \mathrm{d}r_\perp^2\, \frac{\omega_\mathrm{p}^2}{v_\mathrm{T}} \int \mathrm{d}s \int \mathrm{d}s'\; F_\mathcal{L}^*(s')\, \boxed{\Upsilon_\alpha \left[\hat{\xi}_p(s,s')\right]}\; E_\mathcal{L}(s) \qquad (\alpha = 2\delta_{L,0})$$

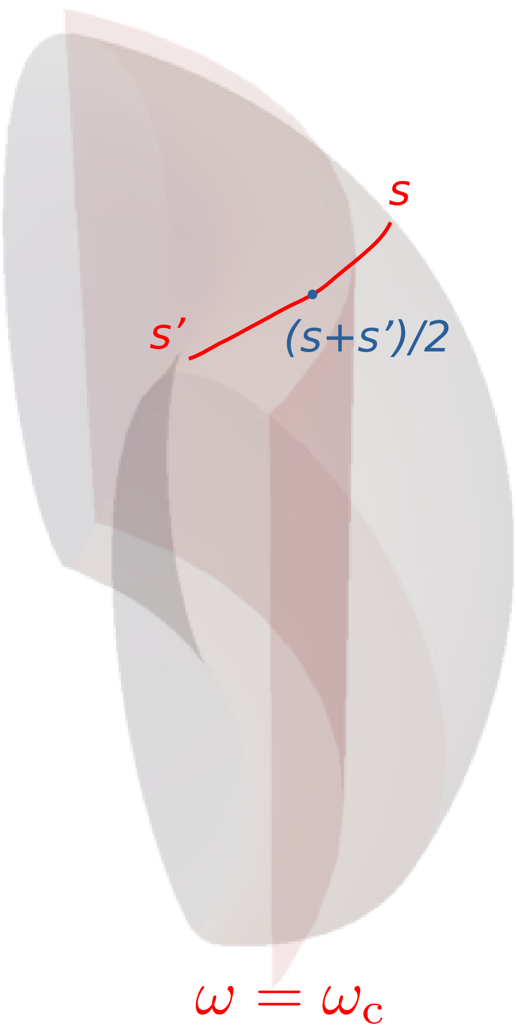
- Crux of the derivation: series of PKDF of 3 variables sum to KDF of a single variable, localized onto a field line.
- These are the same KDFs as for infinite homogeneous plasmas!
- Here, their argument is evaluated at the mid-point between field (*s*) and test (*s'*) points:

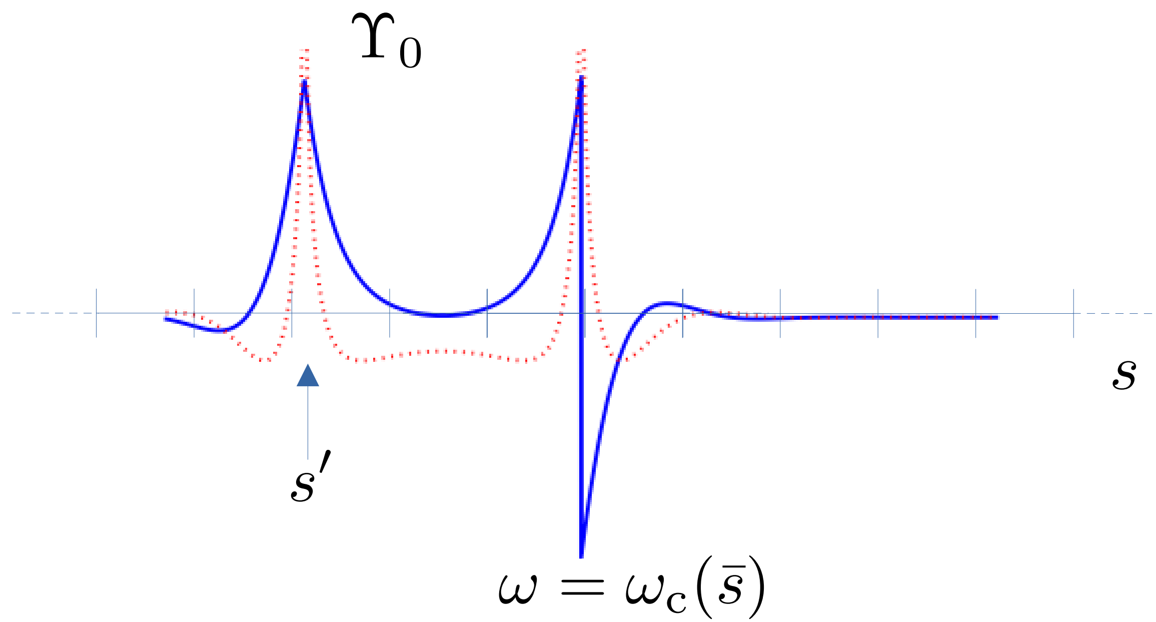
$$\boxed{\hat{\xi}_p(s,s') = \frac{\omega - p\omega_\mathrm{c}(\frac{s+s'}{2})}{v_\mathrm{T}}\, \frac{|s'-s|}{2}}$$

- Correspondence between PKDF and KDF:

$$\Xi_\alpha(\chi,\kappa,\xi) = 2 \sum_{\ell=-\infty}^{\infty} [\mathrm{sign}\,\chi_\ell]^\alpha\, \Upsilon_\alpha(\xi\,|\chi_\ell|)\, \mathrm{e}^{-\mathrm{i}\kappa\chi_\ell}, \qquad \chi_\ell = \chi - 2\ell\pi$$

The cyclotron KDF in presence of $\nabla_{//} B_0$



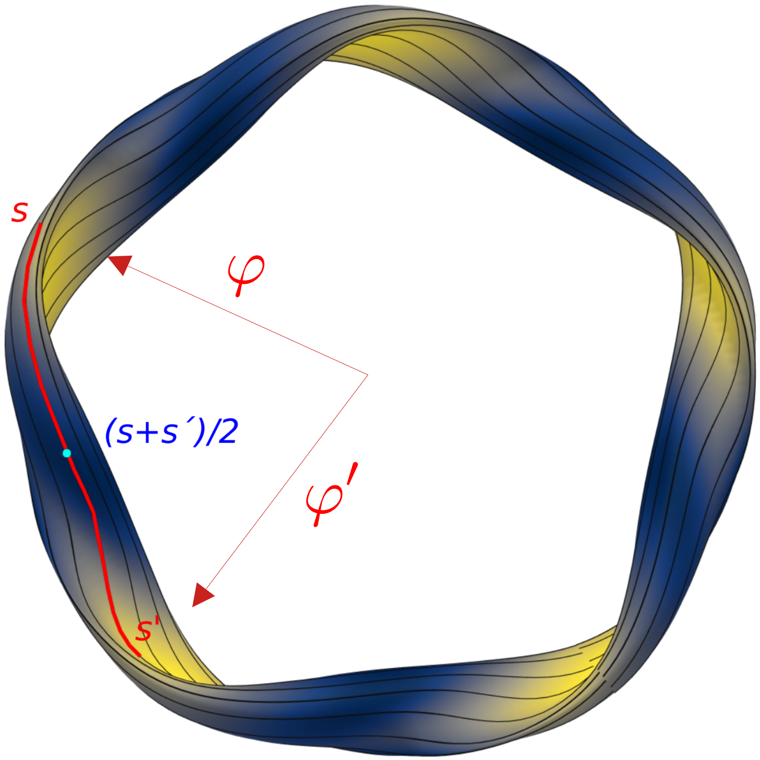


Stellarator: configuration space result

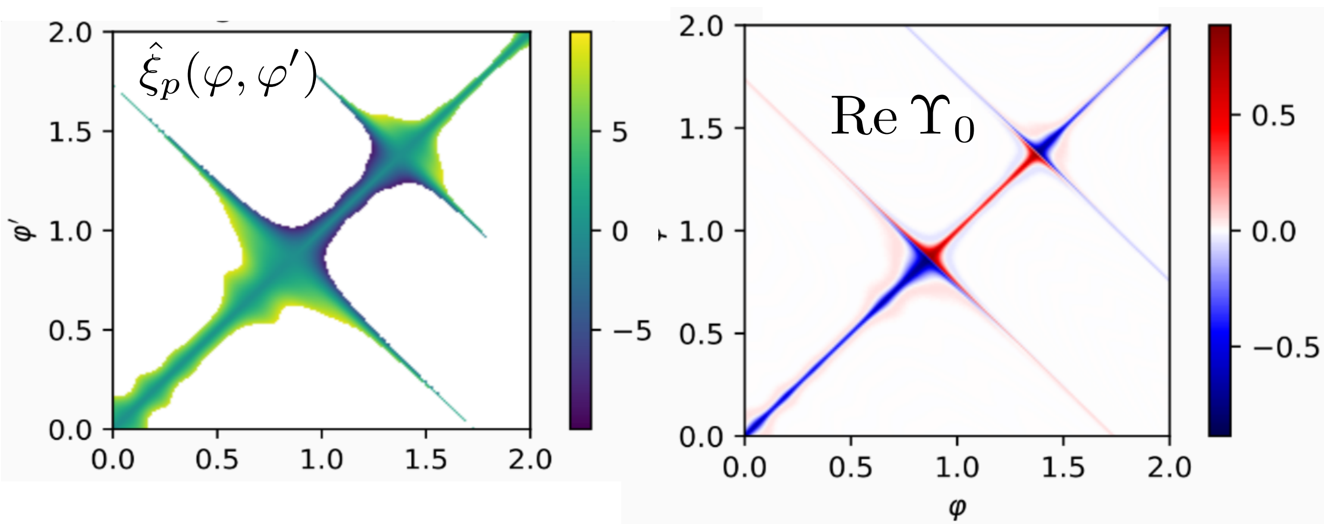
- Same method applied to spectral Maxwellian plasma response of [Vdovin 1996, Fukuyama 2000, Murakami 2006]* $\Rightarrow$ same formal expression as for tokamaks:

$$\mathcal{W}_{\boldsymbol{FE}\beta} = -\frac{\mathrm{i}\varepsilon_0}{2} \sum_{L=-1}^{1} \delta_{L,p}\, 2^{\alpha/2} \int \mathrm{d}r_\perp^2\, \frac{\omega_\mathrm{p}^2}{v_\mathrm{T}} \int \mathrm{d}s \int \mathrm{d}s'\; F_\mathcal{L}^*(s')\; \Upsilon_\alpha \left[\hat{\xi}_p(s,s')\right]\; E_\mathcal{L}(s) \qquad (\alpha = 2\delta_{L,0})$$

$$\hat{\xi}_p(s,s') = \frac{\omega - p\omega_\mathrm{c}\left(\frac{s+s'}{2}\right)}{v_\mathrm{T}}\;\frac{|s'-s|}{2}$$



Cyclotron kernel variation along sample field line:



- Interaction mostly involves close neighbour points, except near 2 cyclotron layers
- Helps determine system sparsity pattern for 3D stellarator geometry

* Ignoring drift waves and specific $\partial f_0/\partial\psi$ effects; and assuming integrable orbits.

FEM implementation

- Following three paths:

  - In-house code, 2.5D slab model (quadratic Nédélec+Lagrange)

  - Psydac (tensor product B-splines): in preparation

  - GetDP/Gmsh-FEM (high degree polynomials): in progress

    PhD started in October 2024,

    Specific goal: enabling / optimizing very large scale computing.

    Linear system preconditioning, domain decomposition and iterative methods (innovative for Maxwell's equations).

- Staged development, initially $\mathrm{FLR}^0 \Rightarrow$ minority & 3-ion ICRH scenarios

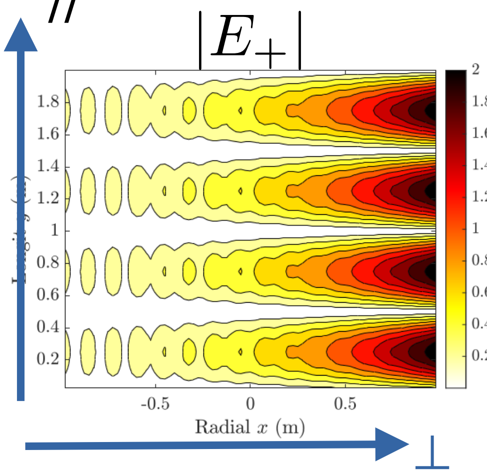
[Reman *et al,* Tuesday]

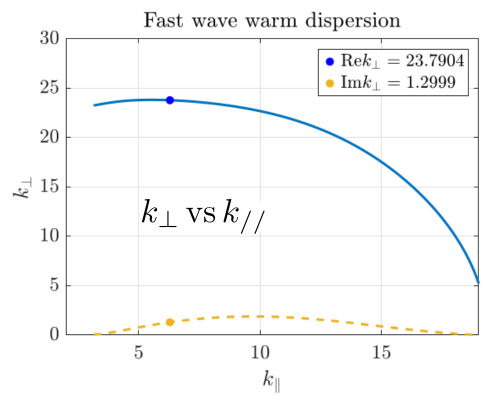
In-house 2.5D code, integral kernel benchmarks

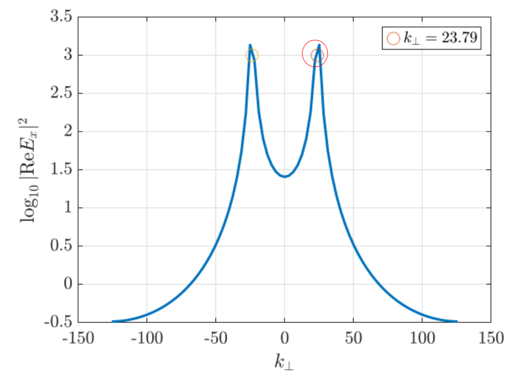
Sine-wave antenna current (51.8MHz, $k_{//}$= 2π m^{-1})

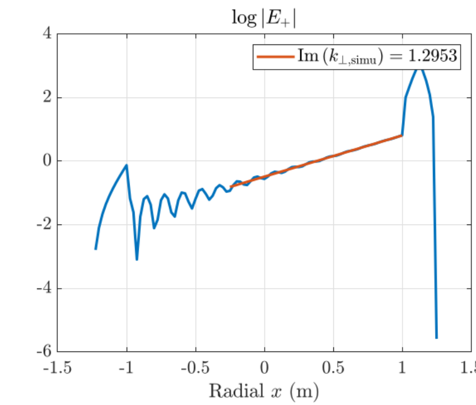
Homogeneous plasma:

Solution matches Re & Im $k_\perp$:



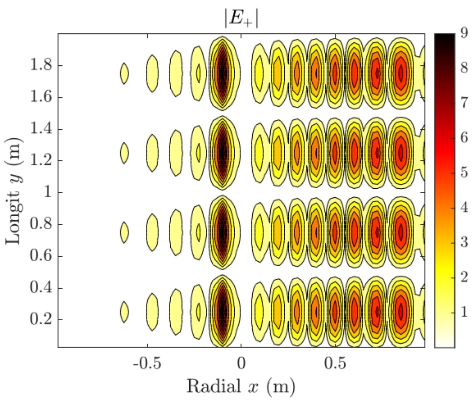


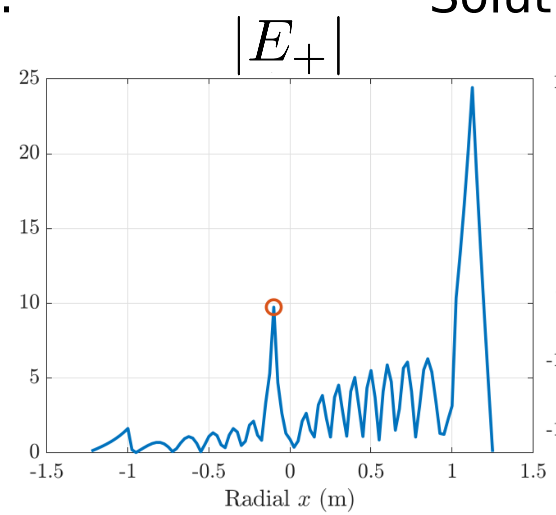


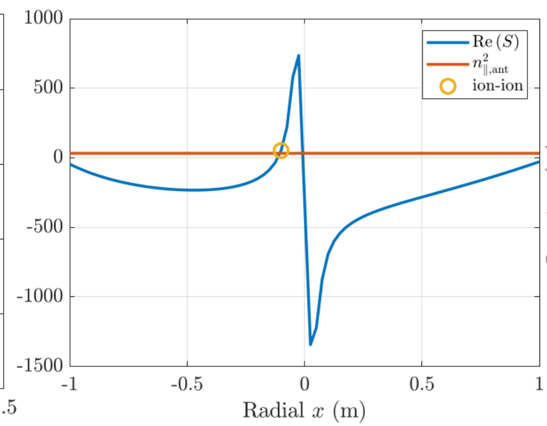


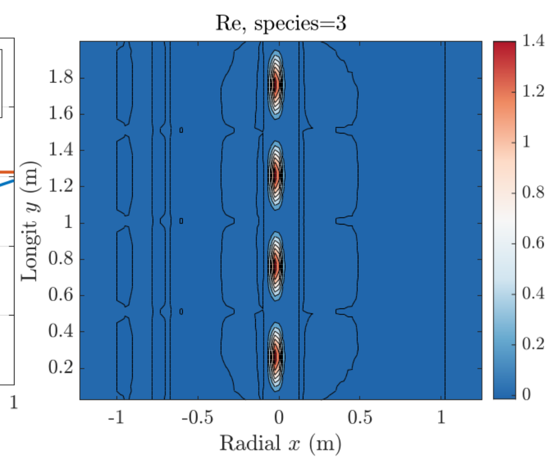
⊥-stratified plasma:

Solution matches ion-ion hybrid layer





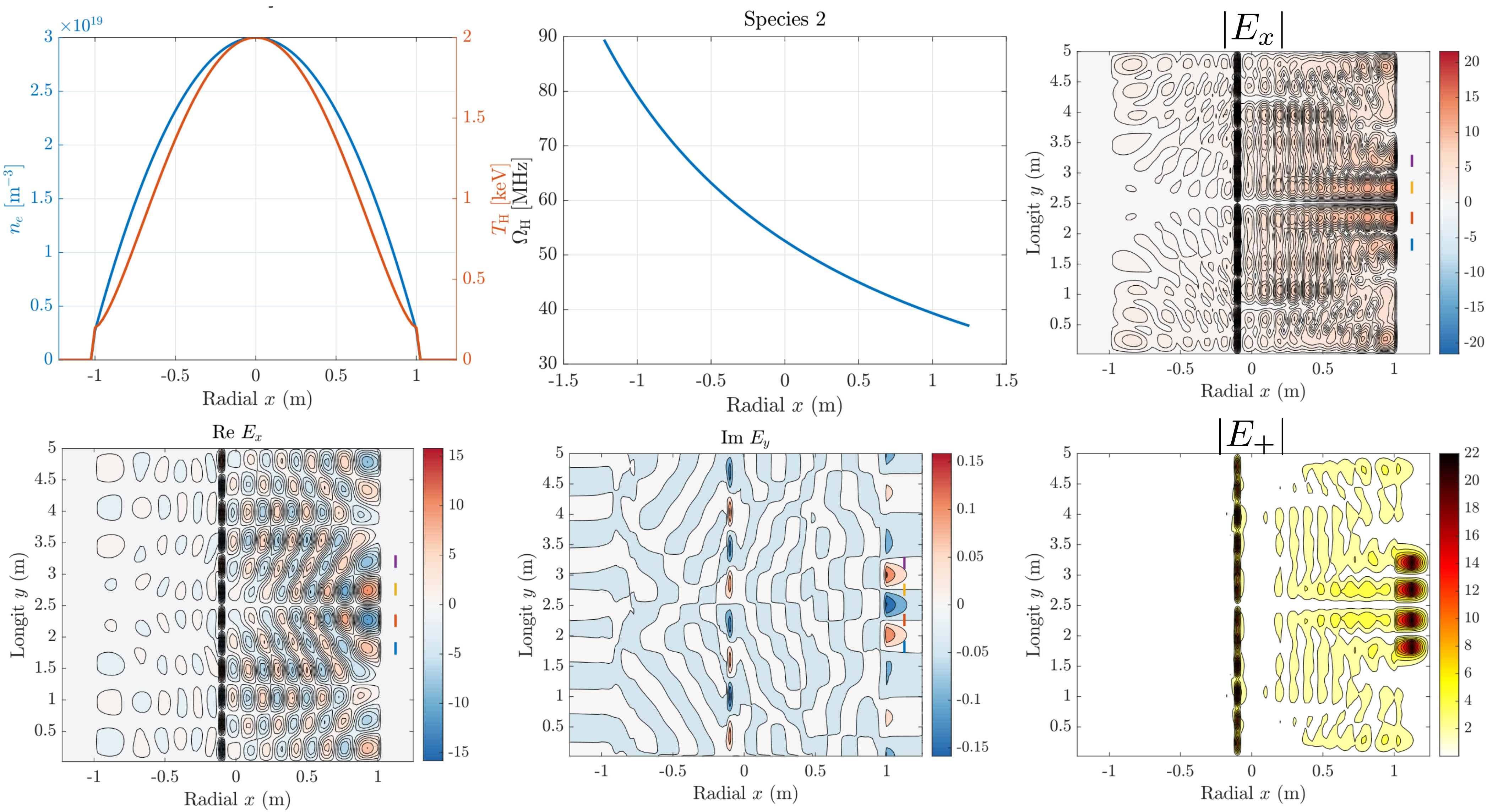




[Reman *et al, Tuesday*]

In-house 2.5D slab code: first results with integral kernel

- JET parameters: 3.45T, 3x10^{19}m^{-3}, 2keV, 5% H in D, A2 antenna (0π0π), 52.6MHz
- Antenna // spectrum captured in a single FE simulation



[Reman *et al, Tuesday*]

Planned development: a graded approach

Toward realistic toroidal geometries:

- Exploit toroidal symmetries *à la* Jaeger 2002
- In-house code: field-aligned mesh in high favour; covariant mapping from FE to physical space will generalize 3D slab.
- More general approach may suit Gmsh-FEM and Psydac

- Code optimization ‘by all means’
- Essential to develop local mesh refinement / auto-adaptive meshing in view of future FLR
- Scalability to very large problems: dedicated development in Gmsh-FEM

- Our 3 FEM codes may call for different optimized solutions

Planned development: a graded approach

Priorities for additional physics:

- FLR effects:
  - The above theory is available with FLR 'full-wave' operator expansion
  - Different possible approaches:
    - ◦ Integral approach // & truncated expansion in powers of $\nabla_\perp, \nabla_\pm$: modifies partial differential operator, needs suitable FE basis functions.
    - ◦ Integral approach // & $\perp$ integral operator similar to Machielsen's in general geometry: $\perp$ nonlocality on the thermal LR scale.

- Non-Maxwellian RF response, with consistent QLFP diffusion coefficient.

Closing comment

- The wave-particle interaction theories currently applied to wave heating - including the present one - are highly approximate versions of the plasma dielectric response in tokamak geometry, first written in full generality in [Kaufman 1972].

- Much additional physics remains worth of investigation.

  (See discussion [Van Eester, Tuesday-14])

- In particular, codes still do not provide adequate treatment of Landau interactions in toroidal geometry, despite earlier pioneering work [Faulconer & Evrard 1991, Puri 1992] and criticism [e.g. Lamalle 1997].

Conclusions

- Theory & graded implementation with three concurrent FEM tools under vigorous development.
- Offers a configuration space integral approach to modelling // kinetic effects in toroidal devices, i.e. in presence of poloidal field and $\nabla_{//} B_0$.
- Derives from spectral theory & shares its physics contents, providing complementary viewpoint and specific advantages.
- Two families of integral kernels obtained for Maxwellian plasmas, properties investigated in detail, singular behaviour at rational-*q* surfaces clearly extracted,
- Further progress in physics contents necessary, fundamentals still well worth revisiting.
- Staged development:

  - Basic numerical aspects dealt with, simplification ahead with semi-analytical plasma contributions to system matrix;
  - Next: realistic tokamak & stellarator, code optimization, meshing strategies;
  - Then, in a position to extend to FLR;
  - Add phase decorrelation physics to further limit nonlocality;
  - ...

All references will be provided in the paper

Acknowledgments

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