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FÜR PLASMAPHYSIK

Beam Tracing in Phase Space

Paraxial Description of High-Frequency Wave Beams in Turbulent Plasmas

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Outline

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   - The paraxial WKB method (beam tracing).
   - The wave kinetic equation (WKE).
   - Examples.
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   - Exact solution of the WKE.
   - Paraxial expansion of the exact solution.
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   - Paraxial ansatz for the WKE.
   - Full derivation of phase-space beam-tracing equation with diffusive scattering.
   - Examples.
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Introduction: the paraxial WKB method

- The wave equation: time-harmonic electric field with frequency $\omega > 0$,

$$\nabla \times \big(\nabla \times E^\kappa\big) - \kappa^2 \mathrm{Op}^\kappa(\varepsilon^\kappa) E^\kappa = 0, \qquad \kappa = \frac{\omega L}{c} \gg 1,$$
$$\varepsilon^\kappa(x,N) = \varepsilon_0(x,N) + \tfrac{i}{\kappa}\varepsilon_1(x,N) + O(\kappa^{-2}), \qquad N = ck/\omega,$$

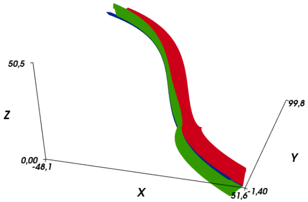
where $\mathrm{Op}^\kappa(\varepsilon^\kappa)$ is the integral operator associated to dielectric tensor ε^κ.

- The paraxial WKB (pWKB) solution[1] (simplified for Gaussian beams)

$$E^\kappa(x) = a^\kappa(x) e^{i\kappa\psi(x) - \kappa\varphi(x)}, \quad \varphi(x) \geq 0,$$
$\varphi(x) = 0$ on a curve $x = x_0(\tau)$,

$$x = x_0(\tau) + y^\alpha \mathsf{e}_\alpha(\tau),$$
$$\psi(x) \approx \psi_0 + N^0_\alpha y^\alpha + \tfrac{1}{2} s_{\alpha\beta} y^\alpha y^\beta,$$
$$\varphi(x) \approx \tfrac{1}{2}\phi_{\alpha\beta} y^\alpha y^\beta, \quad \phi > 0.$$

BT ref. ray and frame



[1] Pereverzev, Nucl. Fusion 32 (1992); Rev. Plasma Phys. (1996); Phys. Plasmas 5 (1998).

Introduction: the paraxial WKB method cont.

- Reformulation in Cartesian coordinates:

$$N^0_\alpha(\tau) = \mathsf{e}_\alpha(\tau) \cdot N^0(\tau),$$
$$\Psi(\tau) = \big(s_{\alpha\beta}(\tau) + i\phi_{\alpha\beta}(\tau)\big)\nabla y^\alpha\big(x_0(\tau)\big) \otimes \nabla y^\beta\big(x_0(\tau)\big)$$

- Beam tracing equations in the pWKB: (with $(H_{xx})_{ij} = \partial^2 H/\partial x^i \partial x^j$ etc...)

$$\begin{cases} \dfrac{dx_0}{d\tau} = \nabla_N H(x_0, N^0), \\ \dfrac{dN^0}{d\tau} = -\nabla_x H(x_0, N^0), \qquad z_0 = (x_0, N^0), \\ -\dfrac{d\Psi}{d\tau} = H_{xx}(z_0) + \Psi H_{Nx}(z_0) + H_{xN}(z_0)\Psi + \Psi H_{NN}(z_0)\Psi, \\ H(z_0) = 0, \quad \Phi\nabla_N H(z_0) = 0, \quad S\nabla_N H(z_0) + \nabla_x H(z_0) = 0. \end{cases}$$

where H is the geometrical optics Hamiltonian of the considered mode.

- One can prove that the constraints restrict the initial conditions only.

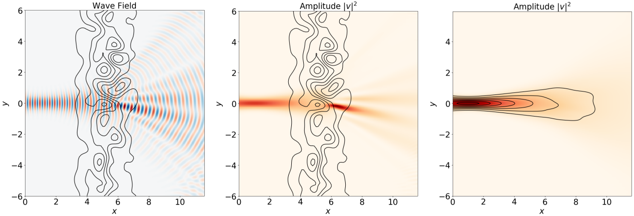
Introduction: the wave kinetic equation

- Fluctuations are represented by a *time-independent* random field μ [2]

$$\nabla \times (\nabla \times E^\kappa) - \kappa^2 \mathrm{Op}^\kappa(\varepsilon^\kappa) E^\kappa = 0, \qquad \kappa = \frac{\omega L}{c} \gg 1,$$

$$\varepsilon^\kappa(x, N) = \varepsilon_0(x, N) + \frac{1}{\sqrt{\kappa}} \varepsilon_F(x) \mu(x) + \frac{i}{\kappa} \varepsilon_1(x, N) + O(\kappa^{-3/2}).$$

- Assumptions on the random perturbation:
  - Scaled[3] by $1/\sqrt{\kappa}$ and $\kappa \gg 1$.
  - Spatially non-dispersive (valid away from the resonance).
- Statistically averaged quantities vs samples.



[2] McDonald, Phys. Rev. A 43 (1991).
[3] Ryzhik, Papanicolaou and Keller, Wave Motion 24 (1996).

Introduction: the wave kinetic equation cont.

- Quantity of interests: the two-point field correlation

$$\rho_E^\kappa(x,x') := \mathbb{E}\big(E^\kappa(x)E^\kappa(x')^*\big) = \big(\frac{\kappa}{2\pi}\big)^d \int e^{i\kappa(x-x')\cdot N} W^\kappa\big(\tfrac{1}{2}(x+x'),N\big)dN.$$

- In the semi-classical limit $\kappa \to +\infty$, the *Wigner matrix* W^κ is (formally)

$$W^\kappa = \sum\nolimits_j w_j^\kappa e_j e_j^* + O(1/\kappa),$$

where e_j are the standard geometrical optics polarization unit vectors.

- The *Wigner function* w_j^κ solves the wave kinetic equation (WKE) for beams

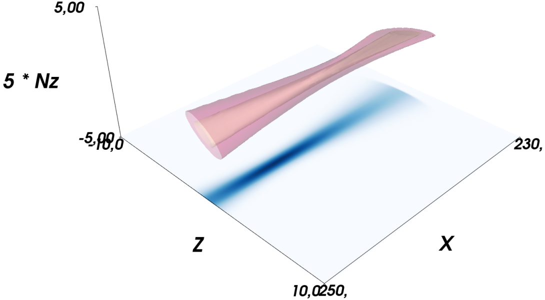
$$\underbrace{H_j w_j^\kappa = 0}_{\text{dispersion relation}}, \qquad \underbrace{\{H_j, w_j^\kappa\}}_{\text{propagation}} = \underbrace{-2\gamma_j w_j^\kappa}_{\text{absorp.}} + \underbrace{\sum\nolimits_l S_{jl}^\kappa(w_l^\kappa, w_j^\kappa)}_{\text{scattering from fluct.}},$$

where $\{f,g\} = \nabla_N f \cdot \nabla_x g - \nabla_x f \cdot \nabla_N g$ is the canonical Poisson bracket. This form of the WKE is implemented in the `WKBeam` code[4].

[4] Weber et al., EPJ Web of Conferences 87 (2015).

Introduction: main idea of this work

- Contours of the Wigner function from `WKBeam` for a focused beam in 2D.



(Four coordinates (x, z, N_x, N_z), but $N_x = -\sqrt{1 - N_z^2}$ in free space.)

- The Wigner function is concentrated around a straight line.
- We try to exploit the localization of the Wigner function using the same ideas of pWKB method.

Phase-space beam tracing w/o fluct.: exact solution

- Fixing the mode (and dropping the mode index j), the WKE reads

$$Hw^\kappa = 0, \qquad \{H, w^\kappa\} = -2\gamma w^\kappa.$$

- If $\nabla H \neq 0$ and with suitable boundary conditions, there are canonical coordinates $z = (x, N) = z_c(\tau, h, \zeta)$,

$$\tau = \tau(z), \quad h = H(z), \quad \zeta^\mu = \zeta^\mu(z), \quad \mu = 1, \dots 2d - 2.$$

  Particularly, $\{\tau, H\} = 1$, $\{H, H\} = 0$, and $\{H, \zeta^\mu\} = 0$.

- In the new coordinates

$$hw^\kappa = 0, \quad \partial_\tau w^\kappa = -2\gamma w^\kappa,$$

  and we have the analytical solution

$$w^\kappa(z) = \frac{2\pi}{\kappa}\delta(h) w^\kappa_*(\zeta) \exp\Big[-2\int_0^\tau \gamma_H(\tau', \zeta) d\tau' \Big],$$

  where w^κ_* is determined by boundary conditions and $\gamma_H(\tau, \zeta) = \gamma|_{H=0}$.

Phase-space beam tracing w/o fluct.: reference curve

- For Gaussian beams, we choose the boundary condition

$$w_*^\kappa(\zeta) = W_0^\kappa \exp\Big[-\kappa g_{\mu\nu}\zeta^\mu\zeta^\nu\Big], \quad g = (g_{\mu\nu}) \text{ symmetric, positive-definite.}$$

- Then, as $\kappa \to +\infty$, the solution w^κ is concentrated on the curve

$$\mathcal{O} = \{H(z) = 0\} \cap \{\zeta(z) = 0\} = \{z = z_0(\tau)\},$$

and since $\{H, H\} = 0$, $\{H, \zeta^\mu\} = 0$,

$$\begin{cases} \dfrac{dz_0}{d\tau} \cdot \nabla H(z_0) = 0, \\ \dfrac{dz_0}{d\tau} \cdot \nabla \zeta^\mu(z_0) = 0, \end{cases} \quad \Rightarrow \quad \frac{dz_0}{d\tau} = -J\nabla H(z_0),$$

where J is the canonical Poisson tensor, i.e., $\{f, g\} = \nabla f \cdot J\nabla g$.

- The reference curve $\mathcal{O}$ is necessarily a solution Hamilton's equations

$$\frac{dz_0}{d\tau} = -J\nabla H(z_0) \iff \frac{d}{d\tau}\begin{pmatrix} x_0 \\ N^0 \end{pmatrix} = -\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\begin{pmatrix} \nabla_x H \\ \nabla_N H \end{pmatrix}.$$

Phase-space beam tracing w/o fluct.: paraxial expansion

- Paraxial expansion: heuristically

$$\zeta^\mu(z) = \underbrace{\zeta^\mu(z_0)}_{=0} + (z - z_0) \cdot \nabla \zeta^\mu(z_0) + O(|z - z_0|^2),$$

$$\Rightarrow \quad g_{\mu\nu}\zeta^\mu \zeta^\nu = (z - z_0) \cdot G(\tau)(z - z_0) + O(|z - z_0|^3).$$

where

$$\boxed{G(\tau) = g_{\mu\nu} \nabla \zeta^\mu\big(z_0(\tau)\big) \otimes \nabla \zeta^\nu\big(z_0(\tau)\big).}$$

- A similar but more precise argument yields the paraxial approximation

$$w^\kappa(z) = w_p^\kappa(z) + O\Big(\frac{2\pi}{\kappa\sqrt{\kappa}} W_0^\kappa\Big),$$

where

$$w_p^\kappa(z) = \frac{2\pi}{\kappa} W_0^\kappa \delta\big((z - z_0) \cdot \nabla H(z_0)\big) e^{-\kappa(z-z_0)\cdot G(z-z_0) - Q},$$

$$Q = -2 \int_0^\tau \gamma(z_0) d\tau'.$$

Phase-space beam tracing w/o fluct.: equations

- Properties of

$$G(\tau) = g_{\mu\nu} \nabla\zeta^{\mu}\big(z_0(\tau)\big) \otimes \nabla\zeta^{\nu}\big(z_0(\tau)\big).$$

  - The matrix G is symmetric, positive semi-definite.
  - The null space is exactly two-dimensional and given by

$$\ker G = \operatorname{span}\left(\frac{\partial z}{\partial \tau}\Big|_{\mathcal{O}} = \frac{dz_0}{d\tau}, \frac{\partial z}{\partial h}\Big|_{\mathcal{O}} \right), \qquad \mathcal{O} = \{z = z_0(\tau)\}.$$

  - We have $\frac{d}{d\tau}\nabla\zeta^{\mu}(z_0) = -D^2H(z_0)J\nabla\zeta^{\mu}(z_0)$.

- In summary, the phase-space beam-tracing equations are

$$\begin{cases} \dfrac{dz_0}{d\tau} = -J\nabla H(z_0), \\ \dfrac{dG}{d\tau} = GJD^2H(z_0) - D^2H(z_0)JG. \end{cases}$$

- The matrix G obeys the same equation obtained in the paraxial approach to wave packets[5], but for beams G is necessarily singular.

[5] Graefe and Schubert, Phys. Rev. A 83 (2011).

Phase-space beam tracing w/o fluct.: recovering Physics

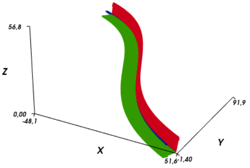
- Two-point correlation requires an integration in N only

$$\rho_E^\kappa(x,x') = \Big(\frac{\kappa}{2\pi}\Big)^d \int e^{i\kappa(x-x')\cdot N} W^\kappa\big(\tfrac{1}{2}(x+x'), N\big) dN.$$

- Under the stronger assumption $\boxed{\nabla_N H \neq 0}$, we can construct an "$x$-adapted frame",

$$x = x_0(\tau) + y^\alpha \mathsf{e}_\alpha(\tau),$$
$$N = N^0(\tau) + y^\alpha \mathsf{f}_\alpha(\tau) + \eta_\alpha \mathsf{e}^\alpha(\tau).$$

PSBT ref. ray and frame



- Then $\rho_E^\kappa(x,x') \approx \rho_p^\kappa(x,x')$,

$$\rho_p^\kappa(x,x') \propto e^{i\kappa N^0_\alpha y_1^\alpha - \frac{\kappa}{4}(\mathsf{A}+\mathsf{D}^s)_{\alpha\beta} y_1^\alpha y_1^\beta} \times e^{-i\kappa N^0_\alpha y_2^\alpha - \frac{\kappa}{4}(\mathsf{A}+\overline{\mathsf{D}^s})_{\alpha\beta} y_2^\alpha y_2^\beta} \times e^{-\frac{\kappa}{2}(\mathsf{A}-\mathsf{D}^a)_{\alpha\beta} y_1^\alpha y_2^\beta},$$

at

$$\begin{cases} x = x_0(\tau) + y_1^\alpha e_\alpha(\tau), \\ x' = x_0(\tau) + y_2^\alpha e_\alpha(\tau). \end{cases}$$

Phase-space beam tracing w/o fluct.: pure beams

- We define $\theta_\mu = (\mathsf{e}_\alpha, \mathsf{f}_\alpha)$ for $\mu = \alpha$, and $\theta_\mu = (0, \mathsf{e}^\alpha)$ for $\mu = d - 1 + \alpha$,

$$\mathsf{G}_{\mu\nu} = \theta_\mu \cdot G\theta_\nu, \qquad \mathsf{G} = \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ {}^t\mathsf{B} & \mathsf{C} \end{pmatrix}, \qquad G = \begin{pmatrix} A & B \\ {}^tB & C \end{pmatrix}.$$

- Then, $\rho_p^\kappa(x, x') = E^\kappa(x)E^\kappa(x')^*$ if and only if G is symplectic, i.e.,

$$\mathsf{A} - \mathsf{D}^a = 0 \iff \boxed{\mathsf{GJG} = \mathsf{J},} \qquad \mathsf{J} = \begin{pmatrix} 0_{d-1} & -1_{d-1} \\ 1_{d-1} & 0_{d-1} \end{pmatrix}.$$

- For wave packets, the whole matrix G is symplectic[6].
- At the launch point, the beam is always pure: conditions on $G(0)$,

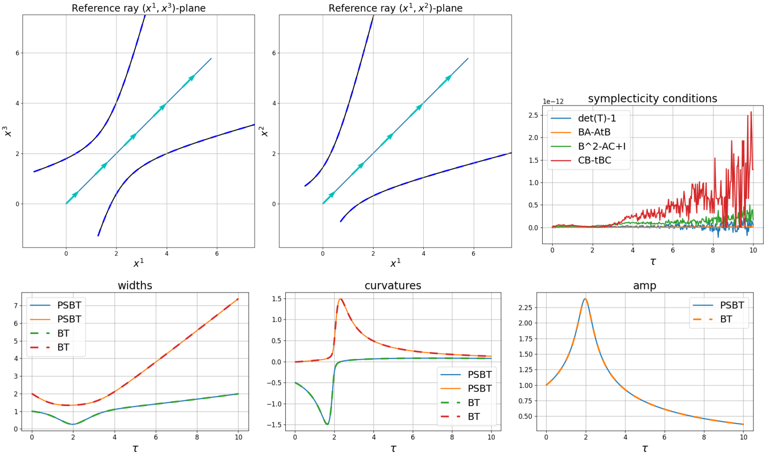
$$\begin{cases} A\nabla_N H - B\nabla_x H = 0, & B\nabla_N H \quad = 0, \\ {}^tB\nabla_N H - C\nabla_x H = 0, & C\nabla_N H \quad = 0, \end{cases} \text{(at the launch point).}$$

- **Conjecture**: Without fluctuations, we should have $\mathsf{GJG} = \mathsf{J}$ for all τ.

[6]Graefe and Schubert, Phys. Rev. A 83 (2011).

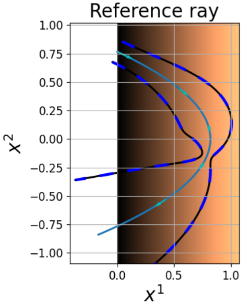
Phase-space beam tracing w/o fluct.: example 1

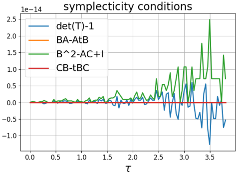
Free space, 3D, fully astigmatic, analytical solution.

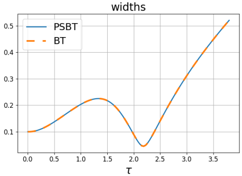


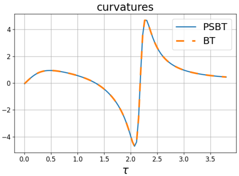
Phase-space beam tracing w/o fluct.: example 2

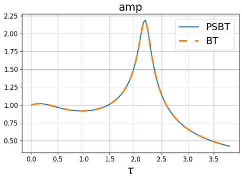
Linear density profile (linear layer), 2D, analytical solution.





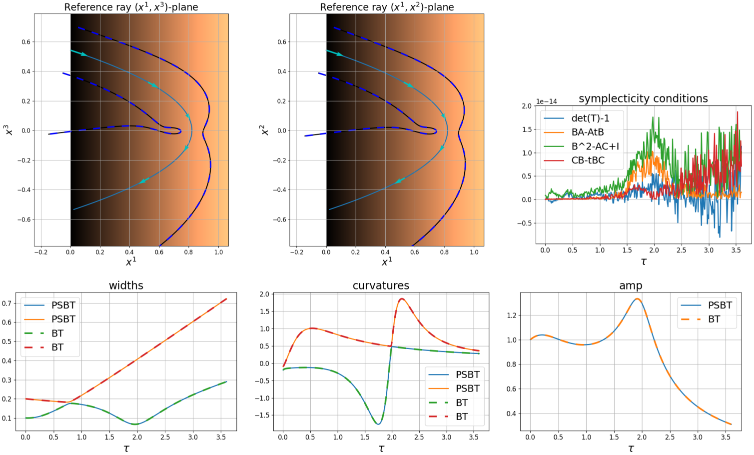






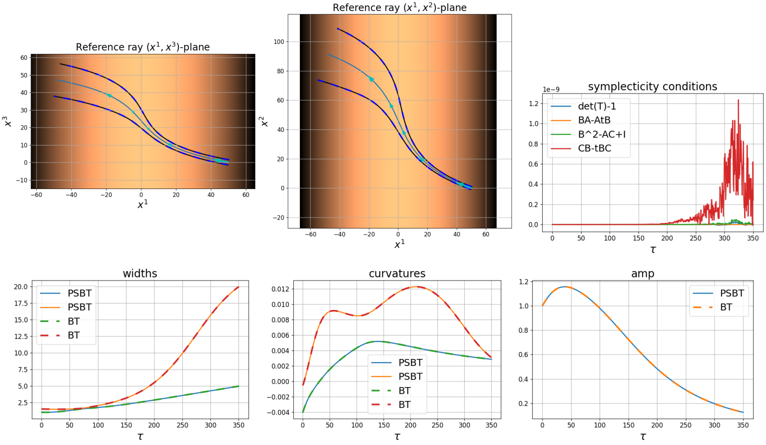
Phase-space beam tracing w/o fluct.: example 3

Linear density profile, 3D, fully astigmatic beam, analytical solution.



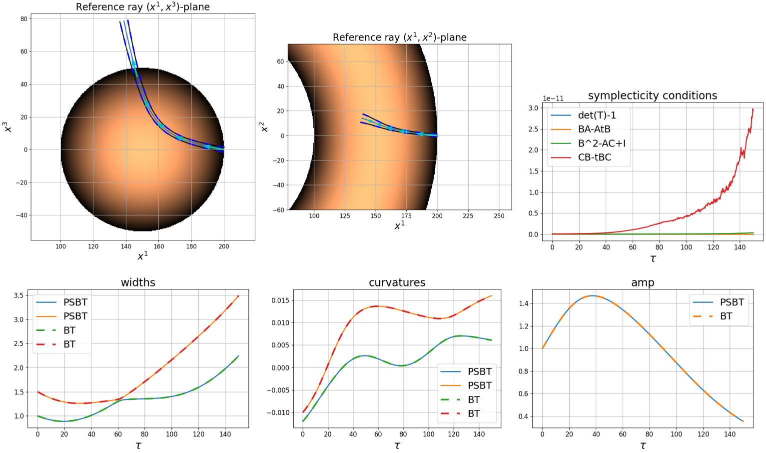
Phase-space beam tracing w/o fluct.: example 4

Isotropic plasma slab, quadratic density profile, 3D, fully astigmatic beam, numerical solution.



Phase-space beam tracing w/o fluct.: example 5

Isotropic plasma torus, quadratic density profile, 3D, fully astigmatic beam, numerical solution.



Phase-space beam tracing w/ fluct.: reformulation of the WKE

- Factor out the singular part of $w^\kappa \propto \delta(H)$,

$$w^\kappa = u^\kappa |\nabla H|^{-1} ds_{\mathcal{M}}, \quad \begin{cases} u^\kappa \text{ smooth function }, \\ ds_{\mathcal{M}} \text{ surface element on } \mathcal{M} = \{H(z) = 0\}. \end{cases}$$

- Equation for the smooth function u^κ:

$$X_H \cdot \nabla u^\kappa = -2\gamma u^\kappa + \tilde{S}^\kappa(u^\kappa) \quad \text{on } \mathcal{M},$$

  with $X_H = -J\nabla H$, and $\tilde{S}^\kappa(u^\kappa)$ is the integral scattering operator.
- Assumptions on fluctuations:
  - Neglect cross-polarization scattering S^κ_{jl} for $j \neq l$.
  - Gaussian correlation function

$$\mathbb{E}\big(\mu(x + \tfrac{s}{2})\mu(x - \tfrac{s}{2})\big) = F(x)^2 e^{-s\cdot\Xi(x)s}.$$

Phase-space beam tracing w/ fluct.: paraxial expansion

- We can write

$$\tilde{S}^\kappa(u_p^\kappa) = \underbrace{\tilde{S}^\kappa_{\text{diff}}(u_p^\kappa)}_{\text{diffusion limit}} + \underbrace{\tilde{S}^\kappa_{\text{res}}(u_p^\kappa)}_{\text{residual}}.$$

- In the diffusion regime (dropping the residual)

$$\begin{aligned} X_H \cdot \nabla u_p^\kappa + 2\gamma u_p^\kappa - \tilde{S}^\kappa_{\text{diff}}(u_p^\kappa) \\ = \Big\{ -2\kappa \Big[\mathcal{T}_1(\tau,\zeta) + g_{\mu\bar{\mu}} \mathrm{d}^{\bar{\mu}\bar{\nu}} g_{\bar{\nu}\nu} \zeta^\mu \zeta^\nu + O(|\zeta|^3)\Big] \\ + \Big[\mathcal{T}_0(\tau,\zeta) + g_{\mu\nu}\mathrm{d}^{\mu\nu} c_0 + O(|\zeta|)\Big]\Big\} e^{-\kappa\zeta\cdot g\zeta}, \end{aligned}$$

where, in particular,

$$\Delta = \sqrt{2\pi}\frac{|e^*(z_0)\varepsilon_F(x_0)e(z_0)|^2 F(x_0)^2}{|\nabla_N H(z_0)|}\left[\frac{\det K^{-1}(x_0)}{\det \Xi(x_0)}\right]^{\frac{1}{2}} \begin{pmatrix} 0 & 0 \\ 0 & \Delta_{NN} \end{pmatrix},$$

$$\mathrm{d}^{\mu\nu} = \nabla\zeta^\mu(z_0)\cdot\Delta\nabla\zeta^\nu(z_0),$$

with $\Delta_{NN} = (K(x_0)^{-1})_{\alpha\beta}\mathsf{e}^\alpha \otimes \mathsf{e}^\beta$, and $K^{\alpha\beta}(x_0) = \mathsf{e}^\alpha \cdot \Xi(x_0)^{-1}\mathsf{e}^\beta$.

Phase-space beam tracing w/ fluct.: equations

- Solving the WKE within a remainder of $O(1/\sqrt{\kappa})$ requires

$$\mathcal{T}_1(\tau, \zeta) + g_{\mu\bar{\mu}} \mathrm{d}^{\bar{\mu}\bar{\nu}} g_{\bar{\nu}\nu} \zeta^\mu \zeta^\nu = O(|\zeta|^3),$$
$$\mathcal{T}_0(\tau, \zeta) + g_{\mu\nu} \mathrm{d}^{\mu\nu} c_0 = O(|\zeta|).$$

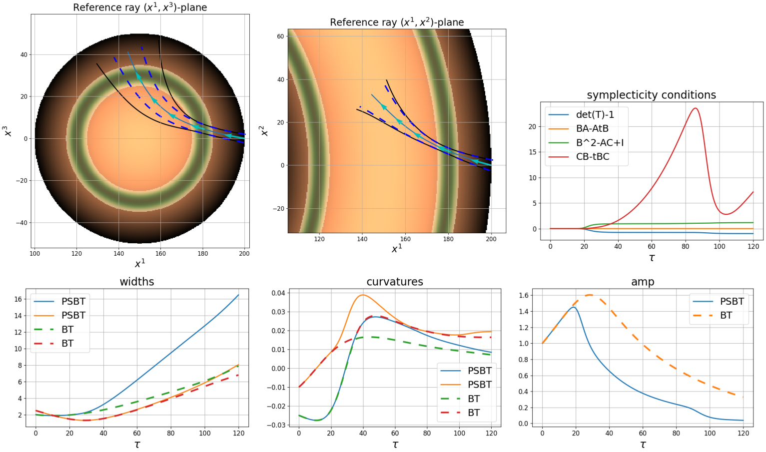
- In summary, we deduce

$$\begin{cases} \dfrac{dz_0}{d\tau} = -J\nabla H(z_0), \\ \dfrac{dG}{d\tau} = GJD^2H(z_0) - D^2H(z_0)JG - 2G\Delta G, \\ \dfrac{dc_0}{d\tau} = -\big(2\gamma(z_0) + \mathrm{tr}\,(\Delta G)\big)c_0. \end{cases}$$

- Initial conditions and reconstruction of ρ_E^κ as before.

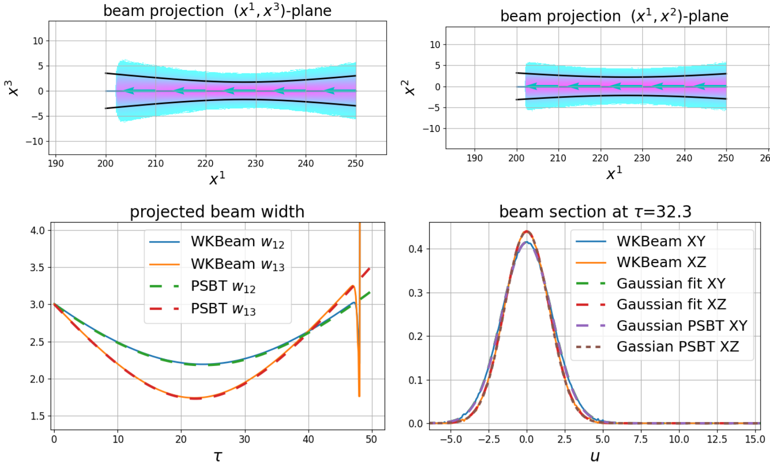
Phase-space beam tracing w/ fluct.: example 0 - scattering effects

Isotropic plasma torus, quadratic density profile, 3D, astigmatic beam, numerical solution.



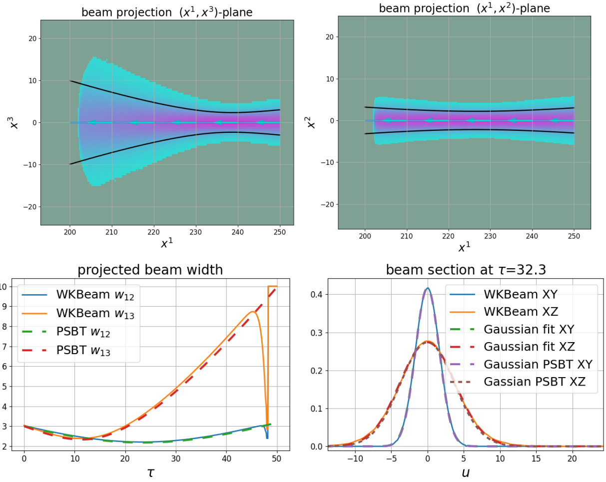
Phase-space beam tracing w/ fluct.: example 1 - no scattering

`WKbeam` benchmark. Free space, 3D, no fluctuations.



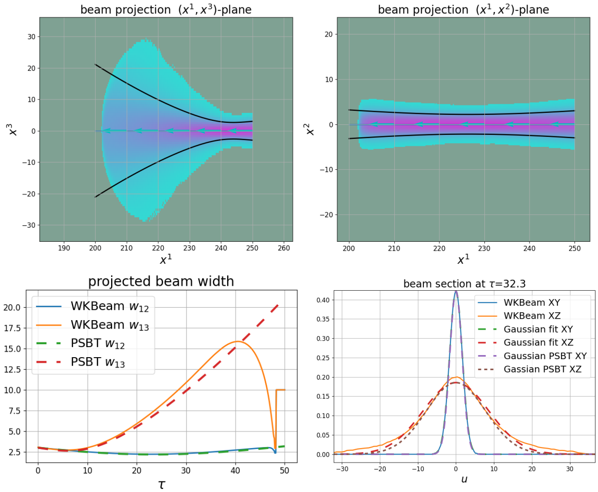
Phase-space beam tracing w/ fluct.: example 1 - diff. scattering

`WKbeam` benchmark. Free space, 3D, diffusive regime $w/L_c < 1$.



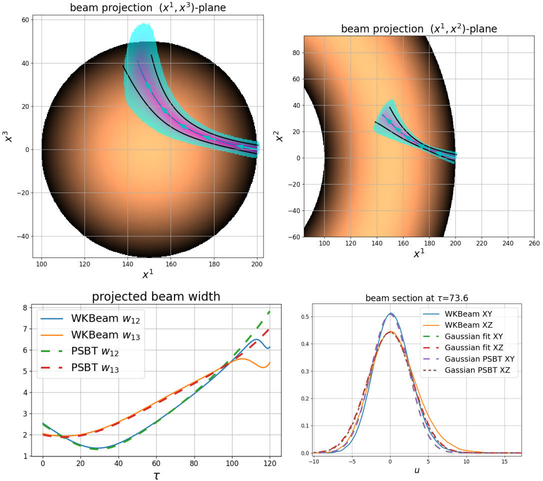
Phase-space beam tracing w/ fluct.: example 1 - non-diff. scatt.

WKbeam benchmark. Free space, 3D, non-diffusive regime $w/L_c > 1$.



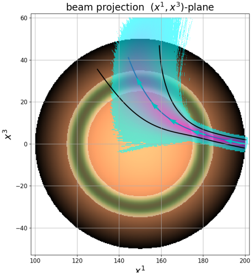
Phase-space beam tracing w/ fluct.: example 2 - no scattering

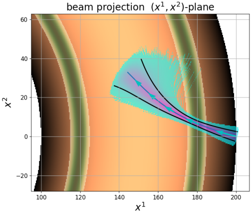
`WKbeam` benchmark. Isotropic torus, 3D, no fluctuations.

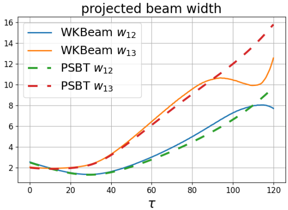


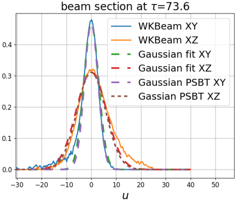
Phase-space beam tracing w/ fluct.: example 2 - diff. scattering

`WKbeam` benchmark. Isotropic torus, 3D, diffusive regime $w/L_c < 1$.



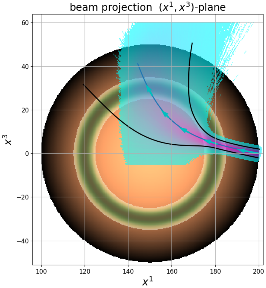


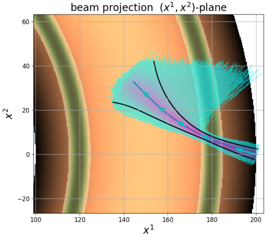


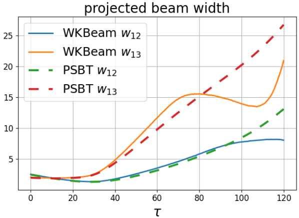


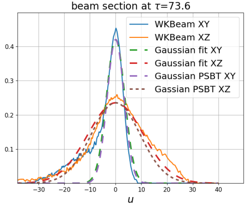
Phase-space beam tracing w/ fluct.: example 2 - non-diff. scatt.

`WKbeam` benchmark. Isotropic torus, 3D, non-diffusive regime $w/L_c > 1$.









Outlook

- We have put forward a beam-tracing technique based on the WKE.
- Reduced symplecticity condition linked to “beam purity” (mixing).
- In a quiescent medium (no fluctuations), we find good agreement with the standard beam-tracing method.
- Diffusive scattering can be accounted for.
- Non-diffusive scattering: ideas in progress[7].

[7] Weber, Ph.D. thesis, University of Ulm (2024) https://doi.org/10.18725/OPARU-52611

Backup slides: evaluation of the numerical error

For the case of example 3.